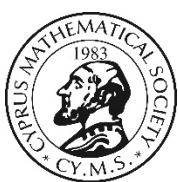


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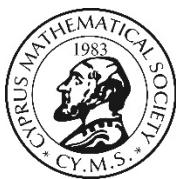
Special Issue: Pythagoras

Volume 21. 2026



CYPRUS MATHEMATICAL SOCIETY
ΚΥΠΡΙΑΚΗ ΜΑΘΗΜΑΤΙΚΗ ΕΤΑΙΡΕΙΑ
Nicosia - Cyprus

Chief Editor
Prof. Gregoris A. Makrides



Mediterranean Journal for Research in Mathematics Education

Special Issue: Pythagoras

**An International Journal
of Cyprus Mathematical Society**

ISSN 1450-1104 Cyprus Mathematical Society

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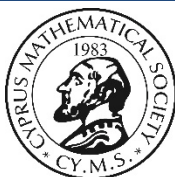
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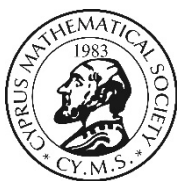
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RESEARCH IN MATHEMATICS EDUCATION**

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PREFACE

This edition of the Journal of the Cyprus Mathematical Society presents a collection of papers originating from presentations delivered at the PYTHAGORAS Conference, held in Cyprus on 3–4 October 2025. The conference was organized by the Cyprus Mathematical Society and the Cyprus Pedagogical Institute and was hosted by the Technical University of Cyprus. Additional organisation collaborated in the event, as listed on the corresponding website.

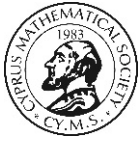
In alignment with its long-standing commitment to promoting the legacy of Pythagoras, the Cyprus Mathematical Society invited several distinguished speakers to contribute to the event. The conference aimed to highlight the enduring impact of Pythagorean thought on mathematics and related fields, to broaden public and academic awareness, and to honour Pythagoras through the installation of a commemorative bust on the grounds of the Technical University of Cyprus in Limassol.

While all papers have undergone editorial review, full responsibility for the content and its scientific accuracy rests with the respective authors.

We extend our warmest wishes to readers and hope they find the contributions both engaging and insightful.

With sincere respect,

Professor Gregoris Makrides
President, Cyprus Mathematical Society



*Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026*

The Mathematical Thought of Archimedes and Apollonius

DEMETRIOS CHRISTODOULOU

Emeritus Professor of Mathematics and Physics
ETH-Zurich

Prologue

- *The mathematical tradition of Ancient Greece originates from Samos, from Pythagoras.*
- *We then had the school which he founded in South Italy, the School of the Pythagoreans.*
- *From there the torch was passed to the School of Athens with Theaetetus and Eudoxus, and from there to the School of Alexandria with Euclid. The mathematical tradition of Ancient Greece culminated in Archimedes and Apollonius, the works of who is the subject of my talk.*

Description

Archimedes

- Area of a spherical segment
- The center of gravity of a parabolic segment
- Hydrostatics

Apollonius

- The theory of focal curves

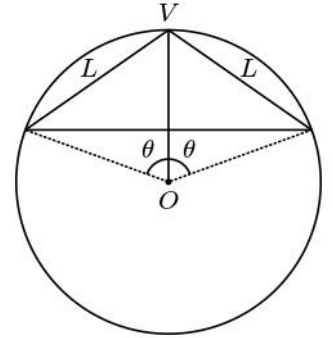
Archimedes

The area of a spherical segment

Finding the area of a spherical segment cut off from a sphere by an arbitrary plane.

Consider a straight line from the center O of a sphere, normal to the disk which constitutes the base of the spherical segment.

This straight line meets the sphere at the vertex V . All points of the circle which is the boundary of the disk have the same distance L from V .



The theorem of Archimedes—last but one of the 1st book of “On the Sphere and the Cylinder”—states that:

the area of the curved surface of a spherical segment is equal to that of a disk of radius L .

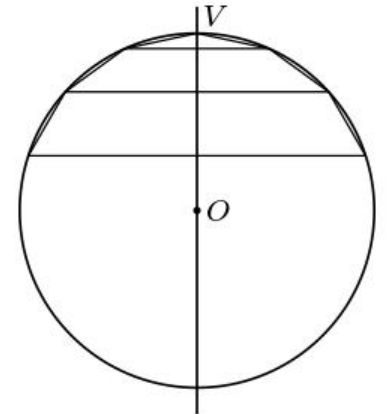
If the segment corresponds to a sector of angle θ and the radius of the sphere is R , we have:

$$L = 2R \sin\left(\frac{\theta}{2}\right)$$

and the area of a disk of radius L is:

$$\pi L^2 = 4\pi R^2 \sin^2\left(\frac{\theta}{2}\right) = 2\pi R^2 (1 - \cos \theta).$$

Therefore, the area of the spherical segment is: $2\pi R^2 (1 - \cos \theta)$.



I shall now describe how Archimedes proves this result.

- He considers a plane which contains the straight line segment OV .
- Such a plane cuts the sphere in a circle and the disk which is the base of the spherical segment in a straight line segment normal to OV .
- The trace of the spherical segment in the plane in question is the segment of the circle cutoff by the straight line segment.
- Archimedes inscribes in this circular segment a polygon with $2n$ equal sides.

- He then considers the surface of revolution which is generated by the polygon when we rotate the plane about the axis OV .
- This surface consists of conical sections, each of which is generated by the rotation of one side of the polygon. We have n such conical sections.
- The area of the surface of a conical section which is generated by the rotation of a straight line segment, here of one side of the polygon, of length s , about an axis, here OV , is:

$$A = 2\pi s \cdot \frac{1}{2}(r_1 + r_2)$$

where r_1 and r_2 are the distances of the end points of the straight line segment from the axis of rotation. Here:

$$s = 2R \sin\left(\frac{\theta}{2n}\right),$$

and for the m th conical section, $m = 1, \dots, n$, we have:

$$r_1 = R \sin\left(\frac{m\theta}{n}\right) \quad r_2 = R \sin\left(\frac{(m-1)\theta}{n}\right).$$

Therefore, the area A_m of the m th conical section is:

$$A_m = 2\pi R^2 \cdot 2 \sin\left(\frac{\theta}{2n}\right) \cdot \frac{1}{2} \left(\sin\left(\frac{m\theta}{n}\right) + \sin\left(\frac{(m-1)\theta}{n}\right) \right),$$

and the area of the surface of revolution generated by the inscribed polygon is:

$$\sum_{m=1}^n A_m = 2\pi R^2 \left(S_n + \sin\left(\frac{\theta}{2n}\right) \sin \theta \right),$$

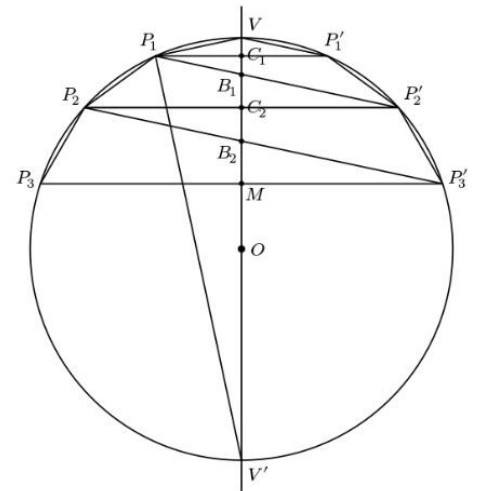
where S_n is the series:

$$S_n = \sum_{m=1}^{n-1} 2 \sin\left(\frac{\theta}{2n}\right) \sin\left(\frac{m\theta}{n}\right).$$

This gives a lower bound for the surface area of the spherical segment.

Archimedes then determines the value of S_n geometrically, in an exceptionally imaginative manner.

Let $P_1, \dots, P_n$ be the vertices of the polygon on the left side of V and $P'_1, \dots, P'_n$ the corresponding vertices on the right side of V .



Let us denote by V' the point diametrically opposite to V . The straight line segments $P_1P'_1, \dots, P_nP'_n$ meet the diameter VV' at the points $C_1, \dots, C_n = M$ respectively.

Let us join P_1 with $P'_2, \dots, P_{n-1}$ with P'_n .

Let $B_1, \dots, B_{n-1}$ be the points where the straight line segments $P_1P'_2, \dots, P_{n-1}P'_n$, respectively, meet VV' .

Let us recall that the angles subtended by equal arcs at different points of the circumference of a circle are all equal.

Therefore, the orthogonal triangles formed are all similar. Hence the ratios of the horizontal segments to the vertical segments are all equal:

$$\begin{aligned} \frac{C_1P'_1}{VC_1} &= \frac{P_1C_1}{C_1B_1} = \frac{C_2P'_2}{B_1C_2} = \frac{P_2C_2}{C_2B_2} = \dots \\ &\dots = \frac{C_nP'_n}{B_{n-1}C_n}. \end{aligned}$$

Consequently, the ratio of the sum of the numerators to the sum of the denominators is also equal.

But since:

$$\begin{cases} P_1C_1 + C_1P'_1 &= P_1P'_1 \\ \vdots & \\ P_{n-1}C_{n-1} + C_{n-1}P'_{n-1} &= P_{n-1}P'_{n-1} \end{cases}$$

the sum of the numerators is:

$$\sum_{m=1}^{n-1} P_m P'_m + C_n P'_n.$$

Also, since:

$$\begin{cases} VC_1 + C_1B_1 &= VB_1 \\ B_1C_2 + C_2B_2 &= B_1B_2 \\ \vdots & \\ B_{n-2}C_{n-1} + C_{n-1}B_{n-1} &= B_{n-2}B_{n-1} \end{cases}$$

the sum of the denominators is:

$$VB_1 + B_1B_2 + \dots + B_{n-2}B_{n-1} + B_{n-1}C_n.$$

But this is simply VC_n . On the other hand, the common value of the ratio is:

$$C_1P'_1/VC_1 = P_1V'/P_1V,$$

something which again follows from the similarity of orthogonal triangles.

We thus conclude that:

$$\frac{\sum_{m=1}^{n-1} P_m P'_m + C_n P'_n}{V C_n} = \frac{P_1 V'}{P_1 V}.$$

Here we have:

$$\begin{aligned} P_m P'_m &= 2R \sin\left(\frac{m\vartheta}{n}\right) \\ C_n P'_n &= R \sin \vartheta \\ V C_n &= R(1 - \cos \vartheta) \\ P_1 V' &= 2R \cos\left(\frac{\vartheta}{2n}\right) \\ P_1 V &= 2R \sin\left(\frac{\vartheta}{2n}\right). \end{aligned}$$

Therefore, the result is equivalent to:

$$\frac{\sum_{m=1}^{n-1} 2 \sin\left(\frac{m\vartheta}{n}\right) + \sin \vartheta}{1 - \cos \vartheta} = \frac{\cos\left(\frac{\vartheta}{2n}\right)}{\sin\left(\frac{\vartheta}{2n}\right)}.$$

Recalling the definition of the sum S_n :

$$S_n = \sum_{m=1}^{n-1} 2 \sin\left(\frac{\vartheta}{2n}\right) \sin\left(\frac{m\vartheta}{n}\right).$$

it follows that:

$$S_n = \cos\left(\frac{\vartheta}{2n}\right) (1 - \cos \vartheta) - \sin\left(\frac{\vartheta}{2n}\right) \sin \vartheta.$$

Let us note that the proof involves the transformation of a sum to another sum which telescopes.

Proceeding, Archimedes considers the limit $n \rightarrow \infty$. Since:

$$\sin\left(\frac{\vartheta}{2n}\right) \rightarrow 0, \quad \cos\left(\frac{\vartheta}{2n}\right) \rightarrow 1$$

we conclude that:

$$S_n = \cos\left(\frac{\vartheta}{2n}\right) (1 - \cos \vartheta) - \sin\left(\frac{\vartheta}{2n}\right) \sin \vartheta \rightarrow 1 - \cos \vartheta.$$

Consequently:

$$\sum_{m=1}^n A_m \rightarrow 2\pi R^2(1 - \cos \vartheta).$$

This is therefore the area of the spherical segment, and the theorem has been proved.

In the preceding description I have simplified things somewhat. The argument of Archimedes is absolutely rigorous. For, he also considers the corresponding circumscribed polygon, the sides of which are parallel to those of the inscribed one.

The area of the surface of revolution generated by the circumscribed polygon, gives an upper bound for the area of the spherical segment.

Moreover, considering different values of n , he shows that every upper bound exceeds every lower bound, that the sequence of lower bounds is increasing, while the sequence of upper bounds is decreasing, and that the difference between the two bounds which correspond to the same value of n tends to zero as n tends to infinity.

And this clenches the proof.

The center of gravity of a parabolic segment

Another work of Archimedes is in the 2nd book of "Plane Equilibria". It is finding the center of gravity of a parabolic segment.

Let Π be the parabolic segment cut off from a parabola by a straight line intersecting the parabola at the points B and B' .

Archimedes calls the straight line segment BB' *base* of the parabolic segment. He considers the point A on the parabola where the tangent is parallel to the base.

This point he calls *vertex* of the parabolic segment. From A he draws a straight line parallel to the axis of symmetry of the parabola. Let O be the point where this straight line intersects the base. The straight line segment AO he calls *diameter* of the parabolic segment, and he calls *height* the length of the diameter.

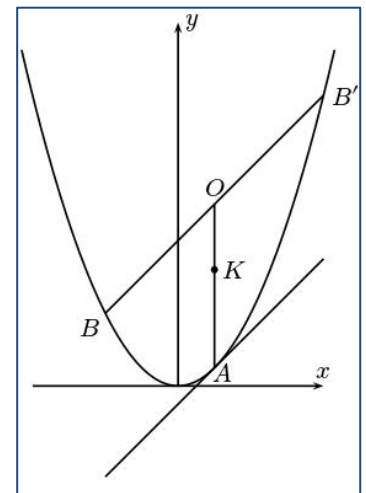


Figure: Parabola $y = x^2/p$

The theorem of Archimedes states that:

The center of gravity of the parabolic segment is the point K on the diameter AO which is defined by the ratio:

$$\frac{AK}{AO} = \frac{3}{5}.$$

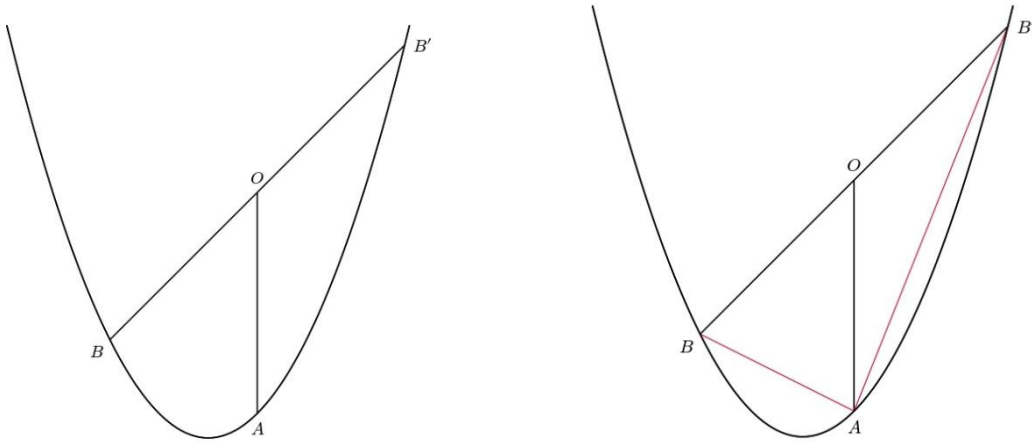
The straight line segment BB' is bisected by the diameter AO .

Every straight line parallel to the base BB' and between the tangent line at A and the base, intersects the parabolic segment in a straight line segment, which, like BB' , is bisected by the diameter AO .

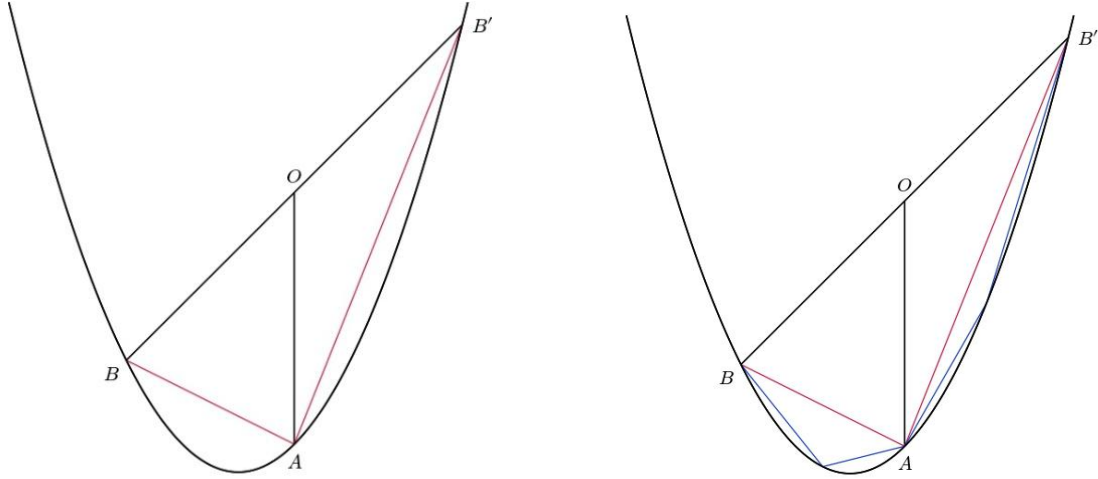
This implies that the center of gravity K lies on the diameter.

Archimedes then shows that the ratio $\lambda = AK/AO$ does not depend on which is the particular parabolic segment under consideration. He succeeds in doing this in an exceptionally imaginative manner.

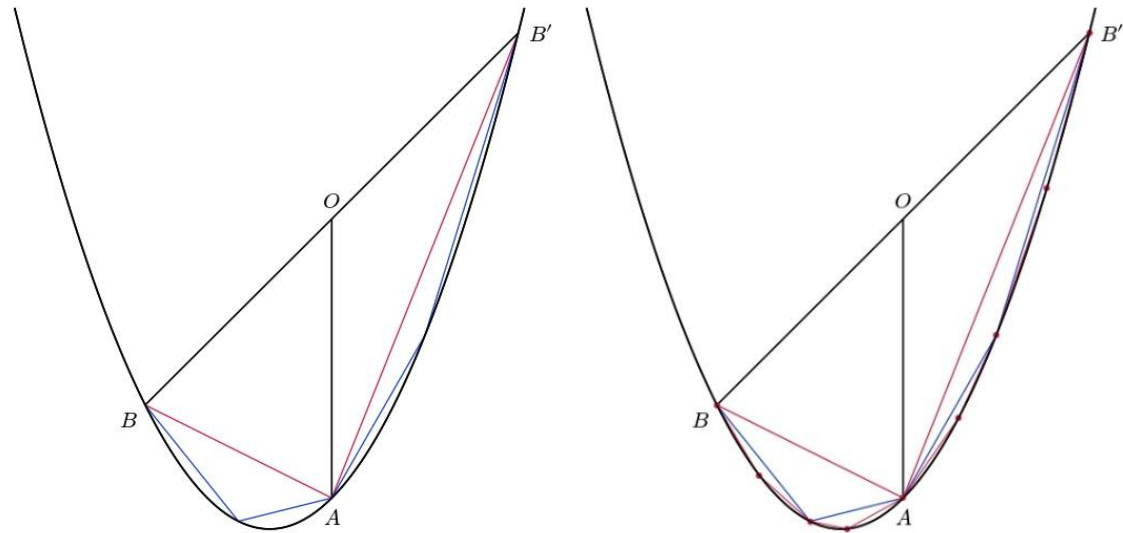
- He defines a process which can have any number n of steps. At step 0 he inscribes in the parabolic segment a triangle with the same base and of equal height.



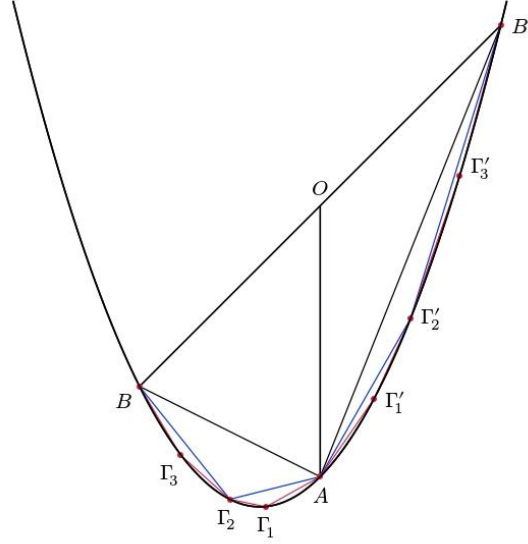
- At step 1 he considers the 2 parabolic segments left over and similarly inscribes in each of these a triangle with the same base as the segment and of equal height.



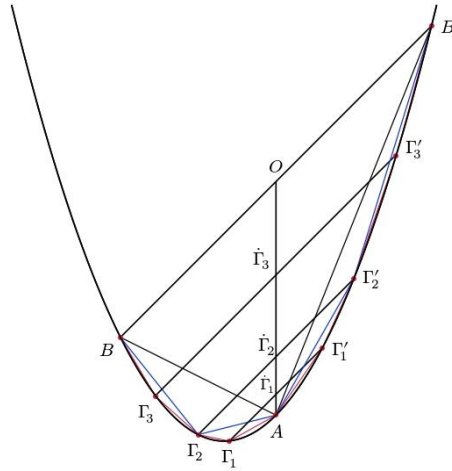
- Proceeding in the same manner, at step n he considers the 2^n parabolic segments left over and inscribes in each of these a triangle with the same base as the segment and of equal height.



- Stopping at step n we have vertices $\Gamma_1, \dots, \Gamma_m$ to the left of the original vertex A in order of increasing distance from A , and corresponding vertices $\Gamma'_1, \dots, \Gamma'_m$ to the right of A , where $m = 2^n - 1$. We thus have an inscribed polygon with $2^{n+1} + 1$ vertices if we include A, B, B' .



- Joining Γ_i with Γ'_i for each $i = 1, \dots, m$, the resulting straight line segments $\Gamma_i\Gamma'_i$ are all parallel to BB' , therefore they are bisected by the diameter AO .



Moreover, and here is the key, these straight line segments cut the diameter AO in the proportions of successive odd numbers, with the number 1 referring to the length which issues from A . That is, if $\hat{\Gamma}_i$: $i = 1, \dots, m$ are the points where the $\Gamma_i\Gamma'_i$ respectively intersect AO , we have:

$$A\hat{\Gamma}_1:\hat{\Gamma}_1\hat{\Gamma}_2:\hat{\Gamma}_2\hat{\Gamma}_3:\dots:\hat{\Gamma}_{m-1}\hat{\Gamma}_m:\hat{\Gamma}_mO = 1:3:5:\dots:(2m-1):(2m+1).$$

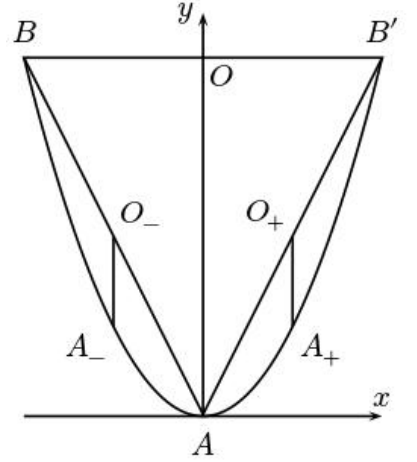
- Using this result Archimedes shows that the center of gravity of the inscribed polygon at step n lies on AO at a distance $\lambda_n \cdot AO$ from A , where λ_n depends only on n .

- Taking the limit $n \rightarrow \infty$ Archimedes then shows that if K is the center of gravity of the parabolic segment, then K lies on the diameter AO and $AK/AO = \lambda = \lim_{n \rightarrow \infty} \lambda_n$ is independent of the particular parabolic segment under consideration.

- To find then the ratio λ it suffices to consider the simplest case where the line segment BB' is orthogonal to the axis of symmetry of the parabola.

- In this case A is the point of intersection of the axis with the parabola and the tangent line to the parabola at A is the straight line through A orthogonal to the axis.

- We denote by Π the original parabolic segment, by T the inscribed triangle, which is isosceles, and by Π_- , Π_+ the two parabolic segments left over, to the left and to the right of T respectively, when T is removed.



- These segments are equal. We denote by A_- , A_+ the vertices of Π_- , Π_+ respectively, and by A_-O_- , A_+O_+ the diameters.
- Let h be the height of Π and h_- , h_+ the heights of Π_- , Π_+ respectively. It follows that:

$$h_- = h_+ = \frac{h}{4}.$$

- Also, the distances of the vertices A_- , A_+ from the tangent line at A are likewise $h/4$.
- We can define rectangular coordinates (x, y) taking A as the origin, The tangent line at A as the x axis and the symmetry axis as the y axis.
- Archimedes then formulates the equilibrium condition which is that the sum of the x -components of the moments of Π_- , Π_+ and T with respect to the origin be equal to the x -component of the moment of Π with respect to the origin.
- Now, the x -component of the moment of a domain in the (x, y) plane is minus the product of the y -coordinate of the center of gravity of the domain with the magnitude of its weight, which is proportional to its area, the direction of the weight being toward the negative z -axis.
- Consequently, the equilibrium condition takes the form:

$$y_-|\Pi_-| + y_+|\Pi_+| + y_m|T| = y_c|\Pi|,$$

where y_- , y_+ , y_m , y_c are the y -coordinates of the centers of gravity of Π_- , Π_+ , T , Π , and $|\Pi_-|$, $|\Pi_+|$, $|T|$, $|\Pi|$ the corresponding areas.

- From the preceding we have:

$$y_c = \lambda h, \quad y_- = y_+ = \frac{1}{4}(1 + \lambda)h.$$

- Also: $y_m = (2/3)h$ the center of gravity of a triangle being the common point of intersection of the three medians. Moreover we have: $|\Pi| = (4/3)|T|$, therefore

$$|\Pi_+| = |\Pi_-| = \frac{1}{2}(|\Pi| - |T|) = \frac{1}{6}|T|,$$

since, as Archimedes demonstrated in his work “quadrature of the parabola”, the area of a parabolic segment is equal to $4/3$ the area of the inscribed triangle.

- Substituting the above in the equilibrium condition and dividing by $h|T|$, this condition takes the form, simply:

$$\frac{1}{12}(1 + \lambda) + \frac{2}{3} = \frac{4}{3}\lambda$$

which gives: $\lambda = 3/5$, and the theorem has been proved.

Hydrostatics

I now come to the most impressive work of Archimedes, his hydrostatics, “On Floating Bodies”, in two books. Archimedes was the first to extend the domain of application of mathematics to the physical world, beyond geometry. He founded the fields of optics, mechanics and hydrostatics.

In mechanics, he introduced the concept of force not only as a vector quantity but as acting along a line. Thus, a force is for Archimedes an oriented line with magnitude, the magnitude of the force. He also introduced the concept of torque, or moment of a force. The definition requires a point of reference which we call origin.

The torque generated by a force, with respect to a given origin, is a vector quantity the magnitude of which is equal to the product of the magnitude of the force with the distance of the line of the force from the origin, and the direction of which is perpendicular to the plane which contains this line and the origin.

The equilibrium conditions are that the sum of the forces vanishes and that the sum of the torques with respect to a given origin also vanishes. When the first condition is satisfied, the sum of the torques is independent of the choice of origin. We have stable equilibrium when any change of situation from

that of equilibrium results in the appearance of a total force or total torque which tend to restore the original situation.

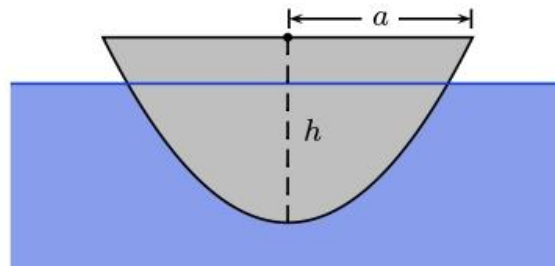
In hydrostatics, when we have a solid partially immersed in a liquid, the free surface of the liquid being a horizontal plane, two forces are exerted on the solid: its weight, which acts downward along a vertical line which passes through the center of gravity of the solid, and the buoyancy which acts upward along a vertical line which passes through the center of gravity of the immersed portion of the solid and has magnitude equal to the weight of the displaced liquid.

But Archimedes did not stop with the formulation of theories. He proceeded to the full development of the theories which he founded solving the most difficult problems which arise in these theories. The solutions of these problems describe phenomena observed in the natural world.

The most impressive work of Archimedes is the 2nd book of "On Floating Bodies" where he studies in every detail the positions of equilibrium and their stability of a solid of given density in the shape of a paraboloid of revolution cutoff by a plane perpendicular to the axis, which floats in a liquid of higher density.

The theorems corresponding to even numbers consider the case where the base of the solid lies above the free surface of the liquid, while the theorems corresponding to odd numbers consider the case where the base of the solid lies below the free surface of the liquid. The first case brings to mind a boat or ship at sea, while the second case brings to mind an iceberg floating in the ocean.

We denote by σ the density of the solid and by ρ the density of the liquid. We assume that $\sigma < \rho$, so that the solid may float in the liquid.



We denote by h the height of the paraboloid and by a the radius of the disk which constitutes the base of the paraboloid. Let p be the length: $p = a^2/h$. Then p is the parameter of the parabola which generates the paraboloid.

Theorem (2)

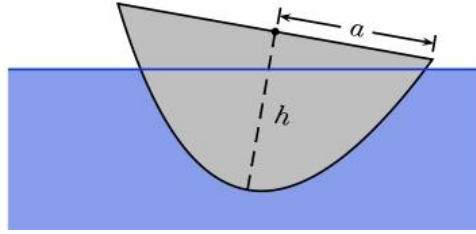
If $h \leq 3p/4$ and the solid is placed in the liquid with its axis inclined at any angle with the vertical, but so that its base does not touch the surface of the liquid, the solid will not remain in this position but will return to the position where its axis is vertical.

Theorem (4)

If $h > 3p/4$ and:

$$\frac{\sigma}{\rho} \geq \frac{(h - \frac{3p}{4})^2}{h^2}$$

the same conclusion holds.

**Theorem (8)**

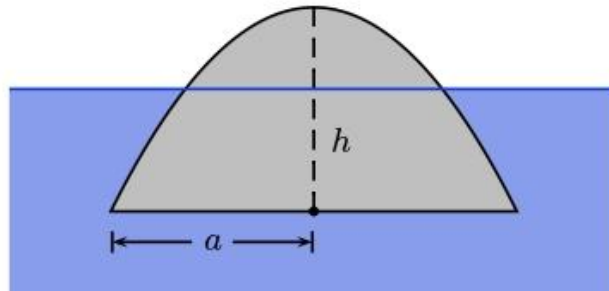
If $h > 3p/4$ but $h < 15p/8$ and: $\frac{\sigma}{\rho} < \frac{(h - \frac{3p}{4})^2}{h^2}$ and the solid is placed in the liquid with its axis inclined at any angle with the vertical, but so that its base does not touch the surface of the liquid, the solid will not return to the position where its axis is vertical and will not remain in any position other than that where its axis makes with the vertical a certain angle, specified in the sequel.

The remaining case, that is $h \geq 15p/8$ and:

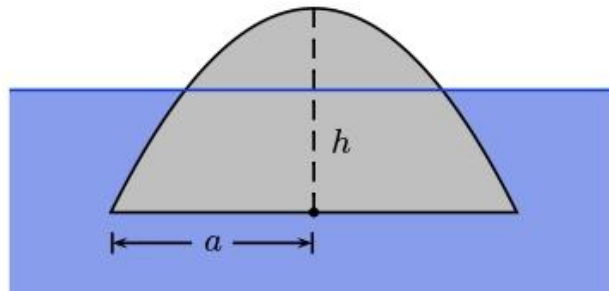
$$\frac{\sigma}{\rho} < \frac{(h - \frac{3p}{4})^2}{h^2}$$

is more complicated, however Archimedes analyzes also this case exhaustively.

In the theorems which correspond to odd numbers the paraboloid is inverted.

**Theorem (3)**

If $h \leq 3p/4$ and the solid is placed in the liquid with its axis inclined at any angle with the vertical, but so that its base is completely immersed in the liquid, the solid will not remain in this position but will return to the position where its axis is vertical.

**Theorem (5)**

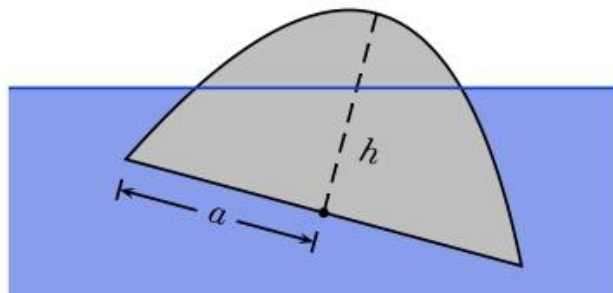
If $h > 3p/4$ and:

$$\frac{\sigma}{\rho} \leq \frac{h^2 - (h - \frac{3p}{4})^2}{h^2}$$

the same conclusion holds.

Theorem (9)

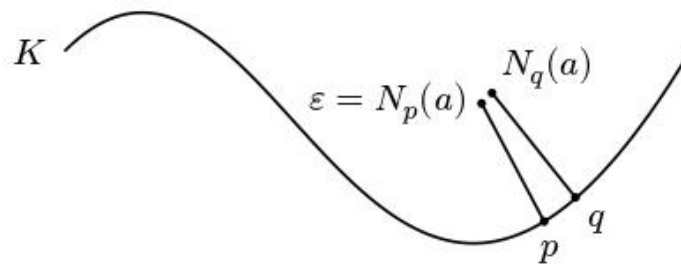
If $h > 3p/4$ but $h < 15p/8$ and: $\frac{\sigma}{\rho} > \frac{h^2 - (h - \frac{3p}{4})^2}{h^2}$ and the solid is placed in the liquid with its axis inclined at any angle with the vertical, but so that its base is completely immersed in the liquid, the solid will not return to the position with its axis vertical and will not remain in any position except that in which its axis makes with the vertical a certain angle, specified in the sequel.



Let us note that in the theorems which refer to the inverted paraboloid the inequalities concerning the ratio of the densities are in the reverse sense.

Apollonius**The theory of focal curves**

The foremost work of Apollonius is the “Conics” and the most impressive part of the “Conics” is the 5th book where Apollonius develops the theory of the focal curves, in particular the focal curves associated to the conic sections: the ellipse, the parabola and the hyperbola.



In general, if K is a curve in the plane and N_p is the normal to K at the point p we denote by $N_p(s)$ the point along N_p at distance s from p . We say that the point $\varepsilon = N_p(\alpha)$ is a focal point along N_p , if, considering points q on K in a neighborhood of p , the ratio of the distance of $N_q(\alpha)$ from ε to the distance of q from p tends to zero when q tends to p .

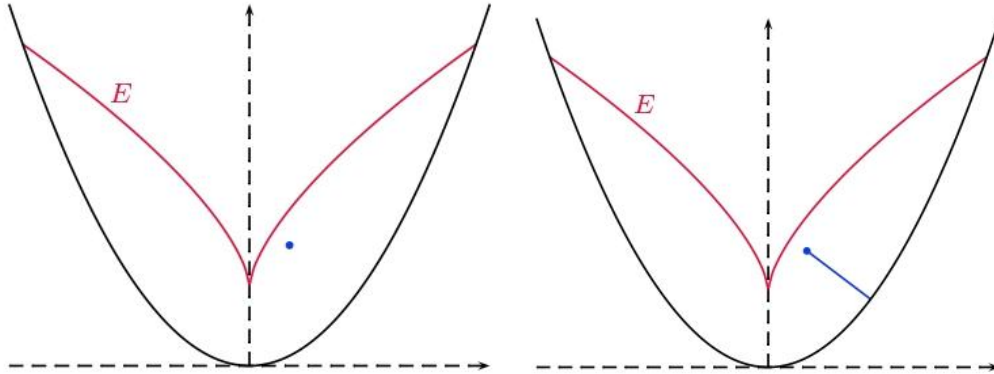
The set of focal points along all normals to K constitutes the focal curve E associated to K .

In the case that K is the parabola: $y = x^2/p$, E is the curve:

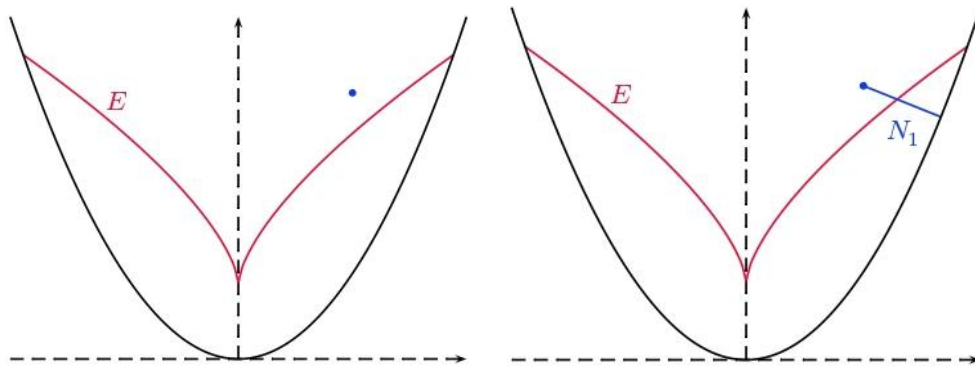
$$x^2 = \frac{16}{27p} \left(y - \frac{p}{2}\right)^3.$$

The theorem of Apollonius regarding the parabola is the following:

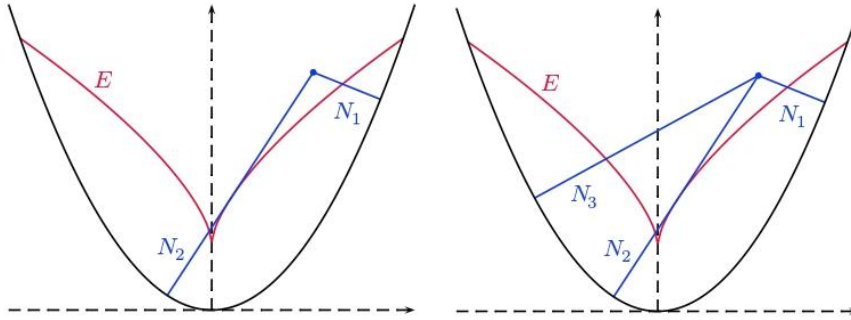
- We are considering the points of the plane which lie above the parabola.
- From a point which lies below the focal curve E only one normal can be drawn to the parabola and the corresponding line segment is globally minimal.



On the other hand, from a point which lies above the focal curve E , three normals, N_1 , N_2 , N_3 can be drawn to the parabola, with N_1 globally minimal,



N_2 which is tangent to E locally maximal, and N_3 locally minimal



Apollonius proved analogous theorems for the other two conic sections, the ellipse and the hyperbola. These theorems, especially the one regarding the ellipse, play an important role in modern science.

Concluding, I shall describe in brief an example of such influence.

The theory of focal curves associated to a curve in the plane extends directly to the focal surfaces associated to a surface in Euclidean space and thence to surfaces in a curved Riemannian manifold.

However, the most interesting extension is to compact spacelike surfaces in a curved spacetime in the framework of the general theory of relativity.

In this case we have two more dimensions, hence two families of future directed null geodesic normals to the surface, an incoming family and an outgoing family.

Penrose introduced in 1965 the concept of a “trapped surface” where an infinitesimal displacement of any element of the surface along each of the two families of null normals results in a decrease of the area of the element.

He then showed that each future directed null geodesic normal to a trapped surface reaches a focal point.

In spacetime, a timelike or null geodesic is the longest curve joining two nearby points on the geodesic.

Therefore, beyond the focal point the null normal, which has zero arc length, stops, in analogy with the theory of Apollonius, being the longest curve which connects the given point with the surface, hence there is a timelike curve, which has positive arc length and connects the point with the surface.

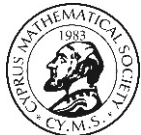
Consequently, such a point does not belong to the boundary of the future of the surface, but to its interior.

This reasoning, in combination with the fact that the surface is compact, leads to the conclusion that the boundary of the future of a trapped surface is compact.

This contradicts the hypothesis that the spacetime arises as the evolution of a non-compact spacelike hypersurface and contains a closed. And the contradiction is lifted only if the spacetime is incomplete, that is if a null geodesic cannot be extended as far as we wish toward the future.

The theorem of Penrose then states that a spacetime which is generated as the evolution of a non-compact spacelike hypersurface and contains a closed trapped surface cannot be complete, it thus comes in time to a certain end.

This theorem gave Penrose the 2020 Nobel Prize in Physics.



Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026

Simone Weil, André Weil, Bourbaki and Pythagorean mathematics

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Quoi de plus proche de nous que la Grèce ? Elle est plus proche de nous que nous-mêmes. Il est douteux que nous ayons une seule idée importante qui n'ait été clairement conçue par les Grecs. (Simone Weil, OC IV 1, p. 64)

ABSTRACT

Simone Weil is one of the most prominent 20th century French philosophers. She is the sister of André Weil, the renowned mathematician, the father of modern algebraic geometry and the initiator of the Bourbaki group. Simone and André Weil shared a love for literature, mathematics, science and philosophy. My aim in this article is to convey, based on their writings and their correspondence, the idea that Pythagoreanism was a central element of their thought. I will put this into context, talking first about the life and work of each of them, showing how much they were linked by essential common ideas, even though their life paths were very different, and how, ultimately, Pythagorean mathematics and philosophy became naturally part of their respective intellectual worlds.

Keywords: Simone Weil, André Weil, Pythagorean mathematics, philosophy of mathematics, mathematics and mysticism.

AMS classification 01A60, 01A20, 00A30

1. Introduction

This article¹ is centered on the role played by Pythagorean mathematics and mysticism in Simone Weil's thinking and writings. To put this in context, I will also talk about the life and work of Simone Weil and about her relation with her brother, the mathematician André Weil. I start here by a few words on Pythagoreanism.

Among a multitude of quotations from modern authors on the influence of the Pythagoreans, let me quote Erwin Schrödinger (1887-1961), from his book *Nature and the Greeks* [17]: “The Pythagorean school is the prototype of a school of thinkers with strongly scientific orientation and at the same time with a well-marked bias, bordering on religious prejudice, towards reducing the edifice of nature to pure reason”. Let me also quote the mathematician and historian of mathematics Michel Chasles (1793-1880). In his treatise *Aperçu historique sur l'origine et le développement des méthodes en géométrie, particulièrement de celles qui se rapportent à la géométrie moderne* (Historical overview on the origin and development of the methods in geometry, in particular those relative to modern geometry) [5], the name Pythagoras is amply mentioned. Chasles writes, on p. 4: “It was primarily to Pythagoras, who incorporated Geometry into his philosophy, and to his disciples, that this science owed its first discoveries.”²

The two quotes have a philosophical taste, and in the pages that follow, I will discuss indeed philosophy and spirituality, together with mathematics. I will try to explain that with Simone Weil, like with the Pythagoreans, notions of harmony, proportion, beauty, rigor, and unity of opposites are no longer specific to mathematics, but govern the spiritual world. The Logos of the Pythagoreans acquires a mediating function, that of the second person of the Trinity.

From a purely mathematical point of view, let me recall that it is generally admitted that Book II of Euclid's *Elements*, which several mathematicians, including André Weil whom we shall quote extensively, regard as a book on geometric algebra, is of Pythagorean origin. Concerning this matter, I refer the reader to the recent article by Negrepontis and Farmaki [10] in which the authors propose a restored Pythagorean version of that book.

The plan of the rest of this article is the following:

Section 2 contains some elements of Simone Weil's biography.

In Section 3, we mention some of the major mathematical achievements of André Weil.

¹ This article is associated with a lecture I gave at the conference “The Life and Contribution of Pythagoras to Mathematics, Sciences, and Philosophy” which took place on October 3-4, 2025 at the Cyprus University of Technology in Limassol. I would like to thank Gregory Makrides, Sotos Voskarides and Christoforos Neophytidis for the invitation and for their care during the conference.

² In this article, all the translations from the French are ours, except when an official translation is indicated in the bibliography.

In Section 6, we review a few comments of Simone Weil on some Pythagorean texts, in which she expresses her belief that Pythagorean philosophical doctrine is close to that of Christianity. In fact, she considered the Pythagoreans as the spiritual ancestors of the Christians.

The final section, §7, is concerned with Pythagorean mathematics as referred to in Bourbaki's *Elements of history of Mathematics*. The relation between Simone Weil, André Weil, and Bourbaki is explained in the previous sections.

2. Simone Weil

There are a number of biographical writings on Simone Weil. The best biography, by far, remains, in my opinion, the one written by her friend Simone Pétrement [15], to whom Simone Weil's parents entrusted the task of writing a biography of their daughter after her death, providing her with many documents they had at their disposal: notes, letters, etc. Other genuine information on Simone Weil's intellectual and spiritual development is found in her published correspondence, especially with her parents and her brother, as well as the latter's writings about his sister, particularly his memoirs published in English under the title *Apprenticeship of a Mathematician*. There is also Simone Weil's correspondence with various people, in particular the volume *Attente de Dieu* (Waiting for God), a collection of letters she sent to Fr. Joseph-Marie Perrin, who was her spiritual mentor, in which she writes at length about her philosophical position with respect to the Catholic Church. This collection includes a long letter to Perrin that was later called *Spiritual Autobiography* and of which we shall quote a passage. Let me also mention the book *Mon dialogue avec Simone Weil* (My dialogue with Simone Weil) written by the same Perrin, published 39 years after Simone Weil's death [14].

In the rest of this section, I will say a few words about Simone Weil which may give a brief overview of her intellectual and spiritual world, before getting to the heart of the matter.

Simone Weil was born in Paris on April 3, 1909 to a father who was a doctor of Alsatian origin and a mother of Russian descent. In addition to her parents, she grew up under the loving and attentive gaze of her brother, André Weil, who was three years older than her and who became later the well-known mathematician. He taught her to read at a very young age. He writes, in his memoirs [28, p. 22]: "I devised a plan to surprise my father for his birthday, April 7: my sister was to read the newspaper to him". André was eight years old, and Simone five. With her brother, she also learned Greek and developed a love of science. We shall say more about her brother later, especially in a section dedicated to him.

In 1928, Simone Weil was admitted to the École Normale Supérieure, in the philosophy section, where she was the only woman in her class.³ At the same time, she was enrolled at the Sorbonne, where she obtained in 1930 a “diplôme d’Etudes Supérieures”, with a dissertation titled: *Science et Perception dans Descartes* (Science and Perception in Descartes). In this work, she analyzes the question of whether science could bring equality or a certain degree of freedom to humankind, or, if, on the contrary, it leads humanity into a new form of slavery. These are themes that she was inclined to keep throughout her life. To her dissertation, she had attached the epigraph ‘Αεὶ ὁ θεὸς γεωμετερεῖ.

During her years of study, Simone Weil was spending a substantial amount of her time and energy in political actions through distributing petitions, gathering money for workers’ union strike and unemployment funds, and social needs. Even though she was firmly aligned with the left, she did not join any political party; the only label that could be applied to her is probably that of being an anarchist.

The period of Simone Weil’s university studies was also a time when Jean-Paul Sartre and Simone de Beauvoir were students. We know that she met them both, but had very little contact with them. Simone de Beauvoir, who was one year older than Simone Weil, relates in her *Memoirs of a Dutiful Daughter* [1] that she sometimes caught sight of the latter in the courtyard of the Sorbonne with the newspaper *l’Humanité* in the pocket of her jacket.⁴ She writes: “She intrigued me because of her reputation of intelligence and her bizarre attire [. . .] A great famine had just devastated China, and I was told that upon hearing this news, she had burst into tears: these tears commanded my respect even more than her philosophical gifts. I envied a heart capable of beating for the entire universe”. A little further, she writes: “One day I managed to tame her. I don’t remember how the conversation started, but she declared in a sharp tone that only one thing mattered in the world today: the Revolution, which would give everyone enough to eat. I replied just as emphatically that the problem was not to make people happy, but to find meaning in their existence. She looked me up and down and said, ‘It’s obvious you’ve never been hungry.’ That was the end of our relationship. I understood that she had labeled me a ‘spiritualist petit bourgeois,’ and it irritated me.” This was the last known interaction between the two would-be philosophers.

³ For those unfamiliar with the French education system, let me recall that the École Normale Supérieure, in Rue d’Ulm, in Paris, also known as ENS Ulm, founded in 1764, is the most difficult university institution to get into in France and probably in the world. Preparing for the ENS entrance exam usually takes two or three years after finishing high school, and studies there generally last four years. Very few students are admitted every year. Until 2005, the school did not award any degree; graduates left with only the title: Ancien Élève de l’École Normale Supérieure (Former Student of the École Normale Supérieure). In 1928, Simone Weil entered the philosophy section of the ENS, at age 19. Her brother had entered, the mathematics section, in 1922, at age 16, which is very exceptional.

⁴ The newspaper *l’Humanité*, founded by Jean Jaurés in 1906, was a left-wing daily from the outset. In 1994, it became the official organ of the French Communist Party.

Simone Weil obtained her agrégation⁵ in 1931. She was assigned to teach philosophy in various high schools in provincial towns: first at the Lycée of Puy-en-Velay, then in Auxerre, Saint Quentin and Bourges. The summer before she began her career as a philosophy teacher, she had taken a trip to Germany, from which she returned frightened by what she had seen of the rise of Nazism and the crushing of the labor movement. It was the year before Hitler came to power. She published several articles on this subject in philosophical journals and also in trade union brochures. Her articles from this period are collected in the volume *Écrits sur l'Allemagne* (1932-1933) (Writings on Germany (1932-1933)) [38].

During her period as a teacher, convinced that access to knowledge should not be a privilege reserved for the wealthy classes but should also be available to the unemployed and workers, Simone Weil used to organize courses and lectures for the latter. Very early on, she began to donate a large part of her salary to unemployment funds. She came into conflict with the administration of her school, which disapproved her social activism and her commitment to defending the poor and the working class. In 1934, she took a leave from her job as a teacher with the intention of becoming a factory worker. Her aim was to physically experience the social oppression, the fatigue of assembly line work, and the constant anxiety of losing one's job that laborers of that era were subjected to. She worked at several factories, always asking to be assigned to the most strenuous jobs. Her health was gradually deteriorating, aggravated by voluntary hunger. During this time, she was publishing articles on social problems, like the status of women in factories, as well as political pamphlets on various subjects, including one on the contradictions in Marxism. At the same time, she was publishing essays on Ancient Greek and Roman history, on the Iliad and on classical Greek theatre, some of them addressed to her colleagues in the factory. She had the conviction that the latter are capable of understanding these tragedies, as much as the educated class.

In 1942, Simone Weil fled France with her parents, with the aim of protecting them, after the Vichy laws were introduced. Her brother André Weil was already in America, where he was offered a job at Lehigh University in Bethlehem (Pennsylvania). After her parents were settled in the United States, she tried to return to France to join the resistance, but she wasn't allowed to because of her poor health. She ended up in England, where she joined the France Combattante.⁶ She was assigned there to an office. She died a few months later, on August 30, 1943, in a hospital in the suburbs of London. She

⁵ The agrégation, in France, is a difficult national competitive exam, in principle for becoming a high school teacher, with higher salary and fewer teaching hours than teachers without agrégation. Practically all students at the Ecole Normale Supérieure take the agrégation. Simone Weil came seventh in the agrégation in philosophy at age 22. Her brother André had taken the agrégation in 1925 (in mathematics) at age 19, and was ranked first. Simone de Beauvoir came second in the philosophy agrégation in 1929, the same year as Jean-Paul Sartre, who came first. It was his second attempt; he had failed the first time. After getting the Agrégation, one is automatically assigned to a high school, unless the person wants to continue with doctoral studies and teach later at the university level. Sartre and Beauvoir, like Simone Weil, became high school philosophy teachers, after obtaining the agrégation.

⁶ The France Combattante was a union of French organisations who were fighting against the Vichy regime and the German occupation of France. The movement was recognized by Great Britain, where Simone Weil was stationed, waiting to turn back to France.

was being treated there for tuberculosis, but the cause of her death was malnutrition.

From a certain point on, Simone Weil became totally immersed in Christian mysticism. From a letter to Father Perrin [33, p. 121]: “Peasants and workers possess that incomparably sweet closeness to God which lies at the bottom of poverty, lack of social consideration and long, slow suffering.” In order not to overstep the bounds of this paper, I will touch only briefly on Simone Weil’s mystical side, even though it is an essential part of her thinking and is closely linked to her deep interest in ancient Greece, Pythagoreanism, and mathematics. I will mention her religious thinking in passing, in relation with Pythagorean mathematics, in Section 6. Readers interested in Simone Weil’s Christian mysticism and its relationship with mathematics can also consult our article [13]. Simone Weil’s mysticism and her ideas on mathematics are also discussed in the article [7] by Laurent Lafforgue.

In her biography, Simone Pétrement recounts that Simone Weil’s mother, responding to a person who was talking to her about her daughter’s glory, said: “Ah! I would have so much preferred that she had been happy!”[15, Vol. 1, p. 20].

It is time to say a few words about André Weil, Simone’s brother.

3. André Weil

André Weil is one of the most important mathematicians of the twentieth century. We have already mentioned his relationship with and influence on his sister. As a child, the least we can say is that he was intellectually precocious. Simone Pétrement, the official Simone Weil biographer, also talks about the brother, based on their mother’s memoirs and other family letters. She reports that in 1914 (André was 8 years old), he came across a geometry book by Emile Borel, began to study it, and was able to solve difficult exercises in the book. With this, the suspicion of his mathematical talent became an evidence [15, vol. 1, p. 26].

André Weil is the main founder of modern algebraic geometry, and he made substantial advances in several other domains. In the lines that follow, I will mention some of the fields in which he made important contributions to mathematics.

The name André Weil is attached to a number of objects and theories that had a profound and durable impact on mathematics. Among these, there are the Borel–Weil–Bott theorem in the representation theory of Lie groups, the Bergman–Weil formula for the representation of holomorphic functions of several variables — an integral formula that generalizes to several variables the fundamental Cauchy integral for holomorphic functions of one variable —, the Shafarevich–Weil theorem in Galois theory, the de Rham–Weil theorem in algebraic topology, Chern–Weil theory and in particular the Chern–Weil homomorphism in the theory of characteristic classes, the Weil–Petersson Kähler metric on Teichmüller space, the introduction of the metaplectic group and a representation of this group, called the Weil representation, the Mordell–Weil theorem in algebraic number theory saying that the group of points on an abelian variety that are rational over a given number field is

always of finite type, and the Taniyama–Shimura–Weil conjecture connecting topology and number theory, which is settled now and which is called the Modularity theorem, of which the Fermat last theorem is a special case. This is only part of the list of mathematical concepts called after André Weil.

I will say no more about André Weil’s work, except to point out that his favourite field was arithmetic, and most of his books and articles are on this topic. Let me mention, without entering in the details, the titles of his doctoral thesis, defended in 1928, *Arithmetic on Algebraic Curves* [20] and that of his 1935 booklet, *Arithmetic and Geometry on Algebraic Varieties* [21]. This book was followed by a note published in Russia, titled *Arithmetic of algebraic varieties* [22]. Let me also mention that in 1950, at the height of his career, André Weil gave a plenary lecture at the ICM in Cambridge, Massachusetts, bearing almost the same title as his article in Russian, *Number theory and algebraic varieties* [23]. I am emphasizing arithmetic here because it was the favorite field of the Pythagoreans, whose motto was that “everything is number” (meaning an integer).

During his stay at the University of Strasbourg (1932–1939), André Weil lectured on arithmetic. It is interesting to read what he says about this, in the *Apprenticeship*: “I taught a course on algebraic number theory. I believe this was the first number theory course to be offered in a French university since the beginning of the century. In the university catalog, it was to be listed as ‘Arithmetic.’ To the dean, this title smacked of primary school; it did not correspond to his notion of the university’s honor, so we changed the course title to ‘Number Theory’ or perhaps ‘Higher Arithmetic,’ and he was happy” [28, p. 111].

It is also important for our purposes here to remember that André Weil is the main founder of the Bourbaki group. Cartier writes in [4]: “Bourbaki was undoubtedly Weil’s creation, even if there was no shortage of strong personalities.” Bourbaki’s *Elements of mathematics* [3] is full of references to the Greeks, in particular the Pythagoreans. It is very probable that a large number of these references are due to Weil. Among the Bourbaki members, he was the one with the most profound knowledge of history. I will quote a few passages on this matter, in §7 of the present article.

It is useful to keep in mind that Bourbaki was founded with the goal of writing a treatise that would be an analogue of Euclid’s *Elements*, first for analysis, and then the project was extended for all mathematics.⁷ Bourbaki shares with the Pythagoreans, in addition to his dedication to mathematics, a cult of mystery: the names of the members were kept secret until after their retirement from the group. Let us also recall that like the Pythagoreans of southern Italy at their peak, Bourbaki wielded at some

⁷ The first members chose as a collective name for the group that of a French general of Greek descent, Constantin Denis Bourbaki (Κωνσταντῖνος Διονύσιος Βούρβακης). The latter, along with other French officers, left for Greece with the permission of the French government to fight the Ottoman occupation after the Greek revolution of 1825 broke on. Bourbaki arrived in Greece around the end of November 1826, leading a Philhellene Corps. He was devoted to Karaïskakis. Makrygiannis mentions him in his *Memoirs* [6]. He died in Greece, on January 27, 1827, at age 40. His name is mentioned on an engraving honoring the Philhellenes on display at the Benaki Museum in Athens (on a staircase). Several members of his family, including his eldest son, Denis, were also fervent Philhellenes. The first name of Bourbaki (the mathematician), Nicolas, was chosen by his godmother, Eveline, the future wife of André Weil. André Weil writes: “My future wife Eveline, who was present at this discussion, became Bourbaki’s godmother and baptised him Nicolas.” [28, p. 81].

point an undeniable occult power in the French academic landscape.

Simone Weil participated in the first Bourbaki congresses; see the photos in Figures 1 and 2 below.

We shall insist on the fact that she understood and retained several issues that were debated during these meetings, and that she later discussed them with her brother.

4. On the relation between the brother and sister

In an interview with André Weil, concerning his relation with his younger sister, and to the question: “Was there a strong intimacy between the two of you?”, he responded: “A very big one. As a child, my sister would spend her time imitating me” [27, p. 10]. Simone Pétrement quotes in her biography [15, Vol. 1, p. 24], from a letter written by Mme Weil to a friend of her about her daughter: “Simone has developed in an incredible way. She follows André everywhere, takes an interest in everything he does, and, like him, now finds that the days are too short. They have an excellent influence on each other: he protects her, helps her climb difficult passages, often gives in to her, and she, who is with him from morning to night, becomes livelier, happier, and more enterprising.” André Weil writes in the Foreword to his book *The Apprenticeship of a Mathematician* [28, p. 11]: “As children, Simone and I were inseparable; but I was always the big brother and she the little sister. Later on we saw each other only rarely, speaking to one another most often in a humorous vein; she was naturally bright and full of mirth, as those who knew her have attested, and she retained her sense of humor even when the world had added a layer of inexorable sadness. In truth we had few serious conversations. But if the joys and sorrows of her adolescence were never known to me at all, if her behavior later on often struck me (and probably for good cause) as flying in the face of common sense, still we remained always close enough to one another so that nothing about her really came as a surprise to me — with the sole exception of her death. This I did not expect, for I confess that I had thought her indestructible. It was not until quite late that I came to understand that her life had unfolded according to its own laws, and thus also did it end. I was little more than a distant observer of her trajectory.”

In a famous passage of a letter that Simone Weil sent to J.-M. Perrin, published under the title *Autobiographie spirituelle* (Spiritual autobiography) in the volume *Attente de Dieu* [33], she writes about her brother: “At age fourteen, I fell into one of those bottomless despairs of adolescence, and seriously thought about dying, because of the mediocrity of my natural faculties. The extraordinary gifts of my brother, who had a childhood and youth comparable to Pascal’s, forced me to be aware of this. I had no regrets about external successes, but about not being able to hope for any access to that transcendental realm where only truly great men can enter, and where truth dwells. I would rather die than live without it. After months of inner darkness, I suddenly and forever had the certainty that any human being, even if his natural faculties are almost nil, can enter that realm of truth reserved for genius, if only he desires the truth and makes a perpetual effort of attention to reach it. Later, when headaches weighed heavily on what little faculties I possess, causing a paralysis that I quickly assumed would probably be permanent, that same certainty made me persevere for ten years in efforts of

attention that were supported by almost no hope of results.”

In the article [12], we discuss the life and thinking of Simone and André Weil, emphasizing their attachment to India and Indian mysticism, another important aspect of the bond between Simone and André Weil.

We now turn to Simone Weil’s correspondence with her brother.



Figure 1: A group photo taken at the Bourbaki congress at Chançay, 1937. From left to right, André Weil, Henri Cartan, Szolem Mandelbrojt, and Simone Weil standing. André Weil writes in [28, p. 118]: “In the meantime, in September of 1937, Bourbaki was again to meet in Chancay. My sister, still intent on improving her knowledge of mathematics, attended our congress”. Chançay is a village situated in the French department of Indre-et-Loire.



Figure 2: Bourbaki congress at Dieulefit, Septembre 1938. From left to right: Simone Weil, Charles Pisot, André Weil, Jean Dieudonné (sitting), Claude Chabauty, Charles Ehresmann and Jean Delsarte. Dieulefit is a commune in the department of Drôme, in the region called Auvergne-Rhône-Alpes. (southeastern France). The photo was taken for the third anniversary of Bourbaki.

5. Correspondence on mathematics and the Pythagoreans

Several letters exchanged between André and Simone Weil reveal the latter's desire to understand her brother's work, and in particular, his results in number theory. Conversely, André Weil was interested in his sister's comments on his writings. In a letter dated February 17, 1934, after informing her of

arithmetical problems on transcendence, he writes: “I will wait patiently for the development of your ideas on mathematics” [37, p. 529].

When it came to questions about history of mathematics, and in particular on the relationships between Greek, Babylonian, Hindu and other mathematical schools, Simone Weil had her own ideas; she was able to establish logical connections, and her brother was pleased and interested in hearing them. He enjoyed their correspondence, particularly on this topic. In the letter to his wife written from prison,⁸ that we quoted above [28, p. 150], he writes: “[. . .] And then there is my correspondence with my sister; nothing could be easier than this at the moment, since it concerns only the most abstract subjects. Greek mathematics, for example: she has some ideas about that.” Several times, his sister would ask him for mathematical references and writings by historians of mathematics that would allow her to test her ideas. A few times, she enquired about Neugebauer,⁹ the major specialist of Babylonian mathematics, asking for references to his works. The possible transmission of ideas on number theory from Mesopotamia to Greece that she alluded to is accounted for in the chapter on “Protohistory” of Weil’s book *Number theory, An approach through history. From Hammurapi to Legendre* [25]. In a letter dated February 1940, sent to her brother in prison and published in her *Works*, t. VII vol. 1, Simone Weil, after describing to him some passages she read in Neugebauer’s book [11] on Babylonian mathematics, writes, about the latter [37, p. 437]:

Strange people, those Babylonians. Personally, I am not very fond of their abstract thinking—the Sumerians must have been much more pleasant. After all, they invented all the Mesopotamian myths, and myths are much more interesting than algebra. But you must be a direct descendant of the Babylonians. As for me, I believe that God, according to Pythagorean teachings, is a perpetual geometer— but not an algebraist. In any case, I was pleased to read in your last letter that you deny belonging to the abstract school.

André Weil was interested in his sister’s ideas. He wrote to her: “I find your ideas on the role of proportion in the history of Greek thought very appealing and, at the same time, quite plausible,[. . .]” (letter dated March 28, 1940 [37, p. 556]).

I would like to highlight two points in Simone Weil’s passage I just quoted. The first is her reference to God and to Pythagorean teaching. This brings us to the fact that Simone Weil, in her philosophical writings, repeatedly stated that she considered the ancient Greeks, starting with the Pythagoreans, to be the true precursors of Christians. In claiming this, she departs from the official position of the Church, which considers that it was the Jewish religion that heralded the coming of Christ. In Section 6, we shall touch on this question, but to discuss it in depth would take us far from the scope of this article. We discuss this in the article [13].

The second point which I would like to highlight is the sentence: “I was pleased to read that you

⁸ Having refused to fulfill his military service, André Weil was first sentenced to five years in prison. Eventually, he was imprisoned in France from February to May 1940, first in Le Havre and then in Rouen. This is another story, which I discuss elsewhere, see [12] and [13].

⁹ Otto Neugebauer (1899-1990) was a mathematician and historian of mathematics and astronomy. He was a specialist, among other things, of mathematics of ancient Mesopotamia. His name is mentioned several times in the correspondence between André and Simone Weil, and in André Weil’s book *Number theory, An approach through history. From Hammurapi to Legendre* [25].

deny belonging to the abstract school”. This fact goes against a certain reputation of Bourbaki, and of André Weil, of being too formalistic and too abstract. On the same subject, we can quote a passage from a letter written in prison by André Weil to his wife that he mentions in the *Apprenticeship*, dated April 22, 1940 [28, p. 150]: “My mathematical fevers have abated; my conscience tells me that, before I can go any further, it is incumbent upon me to work out the details of my proofs, something I find so deadlly dull that, even though I spend several hours on it every day, I am hardly getting anywhere.”

In another letter, sent by André Weil to his sister, dated March 28, 1940, he writes [37, p. 554-555]: “How do you think [the Greeks] discovered the immeasurable? I remember that there is no sure data on this, and that I saw somewhere an ingenious suggestion, with an infinite descent where the thing could be seen on a square and its diagonal (numerically this gives the following: We have $\frac{\sqrt{2}+1}{1} = \frac{1}{\sqrt{2}-1}$ but if $\frac{a}{b} = \frac{b}{c}$ and a, b, c have a common measure, then the same holds for all the lengths obtained by continuing the progression downwards, which is absurd, since these lengths OB' , OC' , OB'' , OC'' , etc. become as small as we want)” (see Figure 3 here). In fact, it is admitted that the Pythagoreans discovered the irrationality of $\sqrt{2}$, by showing that the anthyphairetic expansion of the diagonal to the side of a square is infinite, an operation which is vaguely related to, but much simpler, than the method of infinite descent (both methods are based on the property that of the set of natural numbers is well-ordered). See the recent article [9]. A little bit further in the letter, André Weil writes: “It is probable that, even for $\sqrt{2}$, people started in this way by something

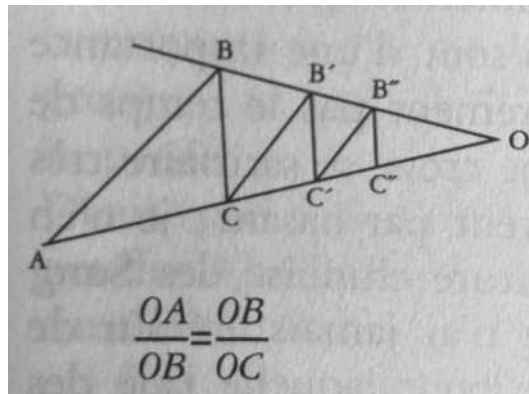


Figure 3: From a letter sent by André Weil to his sister, dated March 28, 1940, commenting on the Pythagorean proof of the incommensurability of the diameter to the side of a square, [37, p. 554-555].

equivalent to the decomposition in prime factors.” He concludes: “In any case, I know of nothing that contradicts the idea (which I no longer remember where I got it from, but which is not my own)¹⁰ that

¹⁰ One must recall that in prison, André Weil did not have his documents or a mathematical library available.

it was the discovery of irrational numbers that led to the first axiomatic research and the first *proofs*¹¹ in the modern sense.”

To André Weil, who was in prison, his sister sent another letter, in February 1940 (with no precise day), in which she writes [37, p. 436]:

Who knows, maybe you will discover some fascinating things? But there is another distraction now that you have free time. I don't know if I mentioned it in the letter I wrote to you in Le Havre, which you must have by now, but never mind. It would be a way of making ordinary people (like me, for example) understand what your current research is all about. I am sure it would be very good practice for you. What do you have to lose? You are not going to waste your time, because you have time to spare. It is all very well to mock those who, like a friend of mine from my days at Rue d'Ulm¹², philosophize about mathematics without knowing anything about it, but perhaps it is precisely mathematicians who should try to do this work. [. . .] What would it cost you to try? I would find it fascinating.

Simone Weil repeatedly urged her brother to write an overview, for non-specialists, of his current research and its significance. André Weil finally responded positively, sending her a long text in the form of a letter dated March 26, 1940, in which he explains his arithmetico-algebraic work (the term he uses). The reflections contained in this letter are of two kinds, he says: some concern the history of number theory, and others the role of analogy in mathematical discovery. It is from this second part that he thought his sister would benefit most.

This letter, which has become a reference text, is published in Simone Weil's correspondence [37, p. 535-552], and also in the Complete Works of André Weil, see [29], text No. [1940a].

Meanwhile, Simone Weil continued to constantly reflect on mathematics and on history of mathematics. In a letter to her brother (who was already in the United States), written in April 1941 (two years before her death), she writes [37, p. 489]: “Can you imagine that recently, while trying to rediscover Archimedes' ‘mechanical method’ for squaring the parabola, I found another method similar to his, which he could very well have used as well!” and she explains her method. In the same letter, she writes: “Can you let me know if Neugebauer has published any new works on ancient mathematics or astronomy since the one I had in my hands (which I believe dates from 1934)?” She writes to him in September 1941 [37, p. 498]: “Reading the Upanishads, I am struck more and more by the similarity between Indian and Greek thought, to the point that I am almost inclined to believe in an Indo-European philosophy that would be the common source; for they are similar in certain respects to the point of identity, without bearing any trace of mutual influence.”

6. On Simone Weil's commentaries on Pythagorean fragments

We already alluded to the fact that Simone Weil considered that the mystical foundations of Christian

¹¹ The emphasis is André Weil's.

¹² École Normale Supérieure, see Footnote 3.

theology can be found in the writings of the Pythagoreans. In her collection of writings that were published under the title *La source grecque* (The Greek source), we find translations and commentaries of several passages of Greek texts, including fragments from the Pythagorean Philolaos.

She translates the fragment

ἔστι ἀρμονία πολυμιγέων ἔνωσις καὶ δίχα φρονεόντων συμφρόνησις

[35, p. 155]¹³ as follows: “Harmony is the unity of a mixture of many, and the unique thought of separate thinkers.”¹⁴ She sees in this fragment a theological connotation. She writes: “It seems to me that the second part of the definition can only apply to a being in several persons,” and she compares this fragment to another fragment of Philolaos (DK58B1),

φιλίαν ἐναρμόνιον ἰσότητα.

which she translates as follows¹⁵ “Friendship is an equality built on harmony”,¹⁶. She declares that the two formulae combined would be a perfect starting point for a theologian who wants to speak about love in the Trinity. A little further, she translates the fragment DK44B11:

ἴδοις δὲ οὐ μόνον ἐν τοῖς δαιμονίοις καὶ θείοις πράγμασιν τὴν τοῦ ἀριθμοῦ φύσιν καὶ τὴν δύναμιν ἰσχύουσιν, ἀλλὰ καὶ ἐν τοῖς ἀνθρωπικοῖς ἔργοις καὶ λόγοις πᾶσι παντᾶ καὶ κατὰ τὰς δημιουργίας τὰς τεχνικὰς πάσας καὶ κατὰ τὴν μουσικὴν.

“We see that the essence and virtue of number do not reign only among religious and divine things, but also in all human actions and relations and in everything that relates to the technique of workmanship and to music.”¹⁷ Commenting these fragments, she writes that they show that the true meaning of early Greek mathematics, which is the foundation of our science, was religious. She declares that this is confirmed by the following passage from Plato’s *Epinomis*, 990d:

[. . . that] which is ridiculously called the measurement of the earth,¹⁸ and which is nothing but the assimilation of numbers not naturally similar¹⁹ to one another, made manifest by the purpose of plane figures. It is evident to anybody who can understand that this marvel is of divine, not human, origin.”²⁰

¹³ In Nicomachus of Gerasa’s *Introduction to arithmetics*, Diels, I, 410, fr. 10.

¹⁴ Her French translation: “L’harmonie est l’unité d’un mélange de plusieurs, et la pensée unique de pensants séparés.”

¹⁵ Diels, 451, lines 12-13.

¹⁶ “L’amitié est une égalité faite d’harmonie.”

¹⁷ “On voit que l’essence et la vertu du nombre ne règne pas seulement parmi les choses religieuses et divines, *mais aussi* dans toutes les actions et relations humaines et dans tout ce qui a rapport avec la technique des métiers et avec la musique.” (The emphasis is Simone Weil’s.)

¹⁸ Plato talks here about geometry.

¹⁹ Non commensurable.

²⁰ “[. . .] ce qu’on appelle ridiculement mesure de la terre, et qui n’est que l’assimilation des nombres non naturellement semblables entre eux, rendue manifeste par la destination des figures planes. Il est évident pour quiconque peut comprendre que cette merveille est d’origine non pas humaine, mais divine.”

She comments on this passage as follows:

“This means that geometry is the science of finding proportional means through the incommensurable proposition (i.e., $\sqrt{m \times n}$), and that it proceeds from a supernatural revelation. This, when considered alongside the passages where Plato describes the mediation between God and man through the image of the proportional mean ($a/b = b/c$) and the numerous passages in the Gospels where Christ uses the same image for his own mediating function (“As my Father has sent me, so I send you, etc.”), makes the invention of geometry in Greece, in the 5th century BC, appear as a prophecy in the strictest sense.” Lafforgue, in his paper [7], recalls that in St Paul’s letter to the Hebrews, the name “Mediator” is explicitly given to Christ (He 9,15 ; 12,24). Simone Weil continues: “Perhaps the least inaccurate translation of the beginning of the Gospel of St. John would be:

In the beginning was the Mediation, and the Mediation was by the side of God, and the Mediation was God. It was in the origin by the side of God. All things came into being through it; without it nothing came into being. In it was life, and life was the light of mankind; and light shone in darkness, and darkness did not overcome it. It was the true light who came into the world, that illuminates every man. And it found itself in the world and the world was born through it, yet the world did not recognize it.”

She writes, a little further: “[. . .] Thus, divine Mediation, through analogical descent, penetrates everything. It unites God to God, God to the world, the world to itself; it constitutes reality in all domains.”

We are back to the central figure of Mediation, that is, Jesus Christ. A mediation allows us to connect different beings and different entities. Christ, for her, is the Mediation between humanity and the Father, and between the Father and the Holy Spirit. She adds that “all this is expressed in the single term ‘logos’, as the name of the second Person of the Trinity.” At several points, she declares that the term λόγος, by which St. John, in his gospel, designates the second Person of the Trinity, is synonymous to Mediation.

7. André Weil, history of mathematics, Bourbaki, formalisation and the Pythagoreans again

André Weil’s contribution to the history of mathematics (and in particular the history of Greek mathematics) is enormous. He had strong ideas on who can write on history, pointing out the sometimes disastrous influence of those who write on the history of mathematics without being mathematicians. Regarding this matter, he writes in [26]: “when a discipline, intermediary in some sense between two already existing ones (say A and B) becomes newly established, this often makes room for the proliferation of parasites, equally ignorant of both A and B, who seek to thrive by intimating to practitioners of A that they do not understand B, and vice versa. We see this happening now, alas, in the history of mathematics. Let us try to stop the disease before it proves fatal.” His book *Number theory. An approach through history. From Hammurapi to Legendre* [25] is both on mathematics and the history of mathematics. In it, we find the development of his idea that ancient number theory is an ancestor of modern algebraic geometry.

We have already talked about the pre-eminent place that André Weil occupied in the Bourbaki group. The latter's volume *Elements of the History of Mathematics* [2] is, broadly speaking, a compilation of the historical notes contained in the various volumes of the whole *Elements of Mathematics*. We shall quote now a few passages from it. Naturally, when discussing the origin of mathematics, Bourbaki gives the exclusive place to the Greeks. At the beginning of the first chapter of his *Theory of Sets* (the first book of the *Elements of Mathematics*), he writes:

Since the Greeks, mathematics has been synonymous with proof; some even doubt that, outside of mathematics, there are proofs in the precise and rigorous sense that the Greeks gave to the word and that we intend to give it here. It is fair to say that this meaning has not changed, for what was a proof for Euclid is still a proof in our eyes; and in times when the concept threatened to be lost and mathematics was therefore in danger, it was among the Greeks that models were sought.

In the André Weil Fund [19], there is a draft of the introduction to this book, in which we find a variation on the passage we just quoted, with some more details. It reads:

There is one mathematics, indivisible: this is the *raison d'être* of this treatise, which aims to explain its elements in the light of a tradition spanning twenty-five centuries. However, since the Greeks, mathematics has meant proof; it is even doubtful whether proofs exist or could exist outside mathematics in the precise and rigorous sense that the Greeks gave to the word and that we intend to give it here. We can rightly say that this meaning has not changed, for what constituted a proof for Euclid is still a proof in our eyes; and in times when the concept threatened to be lost and mathematics was therefore in mortal danger, it was among the Greeks that its paradigms were sought.

The expression “one and indivisible” has obvious theological connotations.

Bourbaki considered the return to the axiomatic method to be precisely a return to the Greeks. He writes in the *Elements of the History of Mathematics* [2, p. 10]:

The essential originality of the Greeks consists precisely in a conscious effort to arrange mathematical proof in such a sequence that the passage from one link to the next leaves no room for doubt and commands universal assent. That Greek mathematicians used, in the course of their research, just like modern mathematicians, heuristic rather than probative reasoning is what Archimedes' treatise on method, for example, would prove (if needed). We will also note, in the latter, allusions to results (found, but not proved) by earlier mathematicians. But, from the first detailed texts known to us (and which date from the middle of the 5th century), the ideal “canon” of a mathematical text is well established. It found later its most complete realization among the great classics, Euclid, Archimedes and Apollonius; the notion of proof, in these authors, differs in no way from ours. We have no text that allows us to follow the first steps of this “deductive method”, which already appears close to perfection at the very moment we observe its existence. We can only assume that it fits quite naturally into the perpetual search for “explanations” of the world, which characterizes Greek thought and which is already so visible among the Ionian philosophers of the 7th century; moreover, tradition is unanimous in attributing the development and refinement of the method to the Pythagorean school, at a time between the end of the 6th and the middle of the 5th century.

The axiomatic method, writes Bourbaki at the beginning of the *Theory of Sets*, “is, strictly speaking, nothing more than the art of writing texts whose formalization is easy to conceive.” We note that the expression “whose formalization is easy to conceive” is not identical to “formal.” It is also important to remember that Weil did make the difference between texts written in a formalized way and those written for research, and this is something that is not sufficiently emphasized by those who speak of him. After discovering a new result, he hated the step where he had to write everything down formally.

We already quoted a passage from a letter to his wife in which he writes that he finds working on the details of his proof to be “deadly dull” [28, p. 150].

We have quoted in §5 a discussion between André Weil and his sister on the incommensurability of the side of a square with its diagonal (that is, the irrationality of $\sqrt{2}$). In the chapter on *Real Numbers* of the *Elements of history of mathematics* [2, p. 185], Bourbaki talks about this result, which he naturally attributes to the Pythagorean school. He writes:

Early Greek mathematics was inseparably linked with speculations, partly scientific, partly philosophical and mystical, on proportions, similarities and ratios, in particular “simple ratios” (expressible by fractions with small numerators and denominators); and it was one of the characteristic tendencies of the Pythagorean school to claim to explain everything by the integer and the ratios of integers. But it was the Pythagorean school, precisely, who discovered the incommensurability of the side of a square with its diagonal (the irrationality of $\sqrt{2}$): the first example, without doubt, of a proof of impossibility in mathematics; the mere fact of asking such a question implies the clear distinction between a ratio and its approximate values, and is enough to indicate the immense gulf that separates Greek mathematicians from their predecessors.

André Weil was convinced that the Greeks had relations with the Babylonians and were familiar with their mathematics, like his sister who had written to him many times on this matter (see the letter of February 1940 already quoted in §5 above, [37, p. 435]). But he also shared with his sister the conviction that the Greeks were not particularly interested in the kind of mathematics the Babylonians developed because what the Greek were looking for was not results but the rigor of the proof. Their discovery of irrationality took on a religious character, because this existence was derived from a rigorous proof. Let us listen to what Bourbaki writes on this subject: In the same *Elements of history of mathematics*, after talking about the fact that the Pythagoreans tried to relate everything to number, Bourbaki writes (p. 35): “[...] Although the discovery of irrational numbers seemed to close this path forever, the reaction it triggered in Greek mathematics was a second attempt at synthesis, this time taking geometry as its basis and incorporating, among other things, the methods of solving algebraic equations inherited from the Babylonians.” Then, on p. 93: “The Pythagorean school, which had rigorously established the concept of commensurable quantities, and attached an almost religious character to it, could not hold to this point of view; and it is possible that it was the failure of repeated attempts to express rationally what led them finally to demonstrate that this number is irrational.”

Let me also recall that Bourbaki traces back the concept of group to the Greeks, and, more precisely, to the Pythagoreans. He writes in the chapter *Groups generated by reflections: root systems* of his *Elements of History*: “Historically, the beginnings of the theory predate the introduction of the concept of groups: it actually has its origins in studies of the ‘regularity’ or ‘symmetries’ of geometric figures, and in particular in the determination of regular polygons and polyhedra (dating back no doubt to the Pythagoreans), which constitutes the crowning achievement of Euclid’s *Elements*, and one of the most admirable creations of Greek genius.”

In closing this section on Bourbaki, I would like to return to Simone Weil, who, as we have seen, participated to several Bourbaki congresses. In her letter to her brother written in February 1940 which we already quoted [37, p. 437], she talks not only about abstract mathematics, but also about concrete problems related to physics. She writes:

“I remember that in Chançay or Dieulefit, you said that these studies on Egypt and Babylon raise doubts as to the role of the Greeks as creators in the field of mathematics, a role that had been attributed to them until now. But I believe that, so far (subject to further discoveries), we find confirmation of this role. The Babylonians seem to have focused on abstract exercises involving numbers, while the Egyptians appear to have proceeded in a purely empirical manner. The application of a rational method to concrete problems and to the study of nature seems to have been unique to the Greeks. The odd thing is that the Greeks must have known Babylonian algebra, and yet we find no trace of it among them before Diophantus; for the algebraic geometry of the Pythagoreans is something quite different. There must be religious concepts underlying this; apparently, the secret religion of the Pythagoreans had to accommodate geometry and not algebra. If the Roman Empire had not destroyed all esoteric cults, we might perhaps have understood something about these enigmas.

Laurent Lafforgue, in his paper [7], rightly emphasizes that neither Simone Weil nor her brother had much regard for algebra used as an automatism. He recalls that in one of her notebooks, Simone Weil wrote, in capital letters, a succinct formula: “MONEY, MACHINERY, ALGEBRA. The three monsters of modern civilization. Complete analogy. [. . .]. The institution of algebra corresponds to a fundamental error concerning the human mind” [30, t. VI, vol; 1, p. 100]. We quoted in §5 Simone Weil’s letter to her brother in which she writes: “I believe that God, according to Pythagorean teaching, is a perpetual geometer — but not an algebraist.” Echoes of all this can be found in Bourbaki’s *Elements of the History of Mathematics* [2, p. 35]:

If we are to accept as authentic the traditional maxim “All things are numbers” of the early Pythagoreans, we can consider it as evidence of an early attempt to reduce the geometry and algebra of the time to arithmetic. Although the discovery of irrational numbers seemed to close this path forever, the reaction it triggered in Greek mathematics was a second attempt for a synthesis, this time taking geometry as a basis and incorporating, among other things, the methods of solving algebraic equations inherited from the Babylonians [. . .]

Such texts could also well be echoes of André Weil’s ideas, reformulated by his sister and returned to him.

Acknowledgement. I would like to warmly thank Bharath Sriraman who read a first version of this paper and kindly sent me several useful comments which helped improving it.

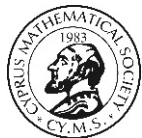
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*Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026*

The Pythagorean Theorem from Euclid to Euler

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December 14, 2025

ABSTRACT

We trace the evolution of Euclid's original formulation of the Pythagorean theorem (Elements, I.47), expressed in terms of squares, through its later reinterpretations: Ptolemy's arithmetical treatment (Almagest), Descartes's arithmetic of line segments (La Géométrie), Newton's analytical use (De Analysi), and Euler's reformulation in terms of trigonometric functions (Introductio in analysin infinitorum).

1 Introduction

There are two formulations of the Pythagorean theorem in Euclid's *Elements*, namely Propositions I.47 and VI.31. The first became part of the foundational knowledge throughout the ages. The latter gained greater prominence in twentieth-century studies of the foundations of geometry and is closely linked to the transformation of Euclid's theory of proportion into the modern framework of metric Euclidean space and the arithmetic of real numbers [BS60].

In this paper, we examine Proposition I.47 together with the underlying theory of equal figures on which its demonstration relies.

We identify an assumption that is not accounted for within this theory and argue that, in this respect, Euclid's argumentation resembles what are now termed dissection proofs.

With regard to Euclid's explicit invocation of the Parallel Postulate in the constructive part of the proof, we show that a weaker assumption suffices – specifically, that all triangles are Euclidean, meaning that the interior angles of any triangle sum to two right angles.

In the following section, we discuss Hindu, Chinese, and Arabic proofs of the Pythagorean theorem as early instances of dissection arguments. We show that these proofs similarly require an assumption guaranteeing the existence of Euclidean triangles and can be recovered in a semi-Euclidean plane.

A decisive stage in the subsequent history of the theorem appears in Descartes's *La Géométrie*. From this point onward, the Pythagorean theorem came to be viewed solely in terms of the sides of a right-angled triangle, without reference to the squares constructed on those sides. We further evaluate the influence of Descartes's reinterpretation on Newton's calculus.

In connection with Newton's *De Analysisi*, and within a brief account of the history of trigonometry, we discuss the arithmetical formulation of the Pythagorean theorem developed by Ptolemy and Heron.

In the final section, we examine Euler's formulation of the theorem in terms of trigonometric functions.

Although Euclid's original proof exhibits certain imperfections, including its reliance on the peculiar proof of Proposition I.4, neither Ptolemy, Descartes, Newton, nor Euler sought to provide an alternative demonstration. Rather, we show that their contributions are best understood as reinterpretations embedded within novel theoretical frameworks.

Thus, we present the evolution of Euclid's original formulation of the Pythagorean theorem in terms of squares, as it is reframed through Descartes's arithmetic of line segments in formulas (1) and (2), Ptolemy's arithmetical treatment in the *Almagest* (4), Newton's analytical application of Descartes's innovation (5), and Euler's trigonometric interpretation (7).

2 Euclid, *Elements*, Proposition I.47

Euclid's Proposition I.47 (ἐκθέσις) reads:

Let ABC be a right-angled triangle having the right angle BAC. I say that the square on (ἀπό) BC is equal to the squares on (ἀπό) BA and AC.¹

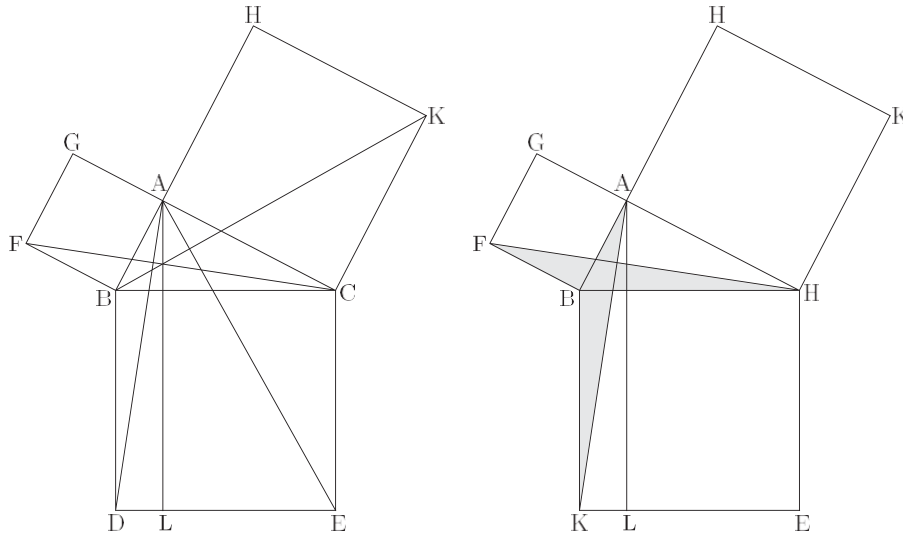


Figure 1: *Elements*, I.47 (left) and a scheme of the proof (right).

¹All translations of the *Elements* follow [Fit07].

The κατασκευή part consists in constructing the squares on the sides of the triangle $\triangle BAC$ – the construction explained in Proposition I.46 – and in drawing through A a line parallel to BD , as well as the lines AD , FC , AE , and BK (see Fig. 1, left).

The ἀπόδειξις proceeds as follows: The triangles $\triangle FBC$ and $\triangle ABD$ are equal by Proposition I.4.

The square $ABFG$ "is double of the triangle" $\triangle FBC$, and the rectangle BDL "is double of the triangle" $\triangle ABD$. Thus, the rectangle BDL equals the square $ABFG$ (See Fig.1)

Similarly, square $ACHK$ equals the rectangle CEL . In conclusion, Euclid writes:

Thus, the whole square $BDEC$ is equal to the two squares GB and HC .

In a modern stylization, it would be presented as follows:

- (1) $\triangle FBC = \triangle ABD$
- (2) $sq\ ABFG = 2\triangle FBC$
- (3) $rec\ BDL = 2\triangle ABD$
- (4) $2\triangle FBC = 2\triangle ABD$
- (5) $rec\ BDL = sq\ ABFG$
- (6) $rec\ CEL = sq\ ACHK$
- (7) $rec\ BDL, rec\ CEL = sq\ BDEC$
- (8) $sq\ ABFG, sq\ ACHK = sq\ BDEC$.

Let us review these arguments in reverse order, from (7) to (1), with (8) as the final conclusion. Finally, labeled (0), we will discuss the construction part of the proof.

Ad 7. There is no textual counterpart for this step, although it is required for completeness. The argument would rely on visual evidence. From the perspective of another instance of such visual evidence, we may say that the rectangle BDL , together with the rectangle CEL , "is the whole square" $BDEC$.² In fact, this is the simplest form of dissection – one that does not involve any manipulation (translation or rotation) of the parts of the dissected figure.

A justification for this step could be grounded in Common Notion 4: *Things that coincide (ἐφαρμόζοντα) with one another are equal to one another*, given that *coincide* in this context means to *match exactly*.

Interestingly, in Proposition II.2 Euclid proves that rectangles such as BDL and CEL are equal to the square they compose. However, that proof employs figures not represented in the diagram, indicating that it proceeds along a different line of reasoning than a dissection [BMP20].

Thus, on the one hand, we regard this step as a footprint of dissection proofs, and on the other, as a precursor of geometric algebra in the setting of Book II.

² See § 4.3 below.

From a historical perspective, in early modern mathematics, Book II was used to motivate algebraic rules such as the distributivity of the product over a sum. In the twentieth century, algebra established its autonomy and adopted rules such as distributivity and commutativity as axioms. In a modern treatment of elementary geometry, since the areas of the rectangles BDL and CEL are ah and bh , respectively, where $h = BD$, $a = DL$, and $b = LE$, the equality (7) follows from the formula:

$$h^2 = h(a + b) = ha + hb,$$

which is obtained by instantiating the distributivity axiom.

Ad 6. Euclid just notes that the reasoning concerning $sq\ ACHK$ and the rectangle CEL proceeds in the same way ($\acute{o}\mu\acute{o}\iota\omega\varsigma$) as that concerning $sq\ ACFG$ and rectangle BDL .

Ad 5. This follows from the application of Common Notion 1 (the transitivity of equality) to (2), (3), and (4), which is not explicitly stated in the text.

Ad 4. According to Heiberg's edition, this is an interpolation [Hei88, vol. I, p. 122]. Nevertheless, it is noteworthy because it reveals the points at which later generations concentrated their scrutiny; it also indicates that, for some commentators, the step does not follow from Common Notion 2 and thus requires an additional rule. In short, whereas modern geometers, following Hilbert, direct their attention to Proposition I.4, ancient and early modern scholars were primarily concerned with the requisite Common Notions.

Ad 2–3. These arguments refer to Euclid's theory of equal figures, developed in Propositions I.35–44, which extends the concept of equality from congruent triangles to non-congruent polygons.

The theory consists of a set of constructions that allow one to transform a polygon ($\epsilon\upsilon\theta\eta\gamma\rho\alpha\mu\mu\omicron\nu$), say A , into a square S . The equality $A = S$ is founded on the Common Notions (hereafter CN) and on Propositions I.1–34, specifically the criteria for triangle congruence and the properties of parallelograms. In Book II, it also employs certain substitution principles, such as: if $AB = CD$, then the square on AB equals the square on CD , among others; see [BMP20].

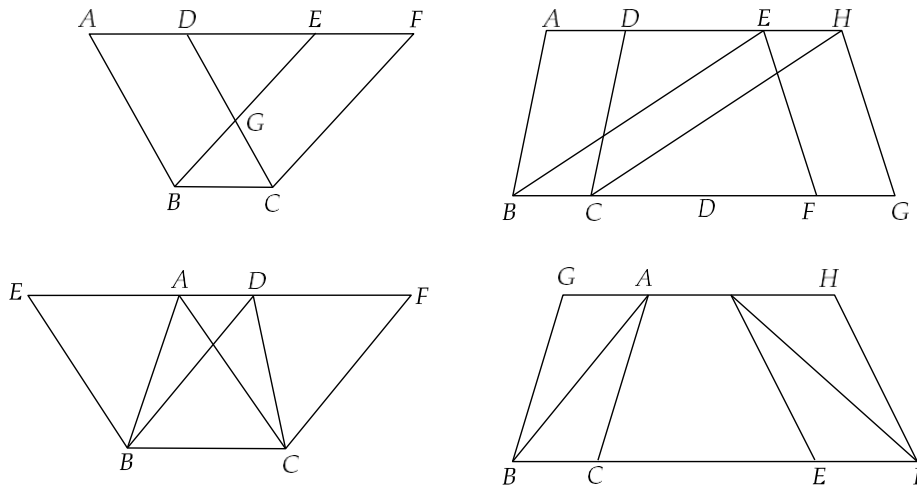


Figure 2: *Elements*, Propositions I.35–38.

It begins with Proposition I.35, which states the equality of the parallelograms $ADCB$ and $EFCB$ that lie on the same base and “between the same parallels,” i.e., have the same height (see Fig. 2, upper-left diagram); specifically, a parallelogram can be transformed into a rectangle.

The proof proceeds as follows: the triangles $\triangle AEB$ and $\triangle DFC$ are congruent. By subtracting $\triangle DEG$, the remaining figures $ADGB$ and $EFCG$ are equal (by CN3). Adding BGC to both $ADGB$ and $EFCG$ then yields that the parallelogram $ABCD$ is equal to the parallelogram $EFCB$ (by CN2).

In Proposition I.36, Euclid shows that parallelograms on equal bases and of the same height are equal; that is $ADCB = EHGF$ (see Fig. 2, upper-right diagram).

Proposition I.37 states that triangles on the same base and of the same height are equal; $\triangle ABC = \triangle DBC$ (see Fig. 2, lower-left diagram). The proof appeals to the equality of the respective parallelograms and to the implicit supposition that *halves of the same thing are equal*. In fact, this amendment is an interpolation [Hei88, vol. I, p. 90] – another item on the list of Common Notions added in the history of correcting Euclid [Hea56, vol. I, pp. 222–224].

Proposition I.38 reiterates this claim for triangles on equal bases: $\triangle ABC = \triangle DEF$ (see Fig. 2, lower-right diagram).

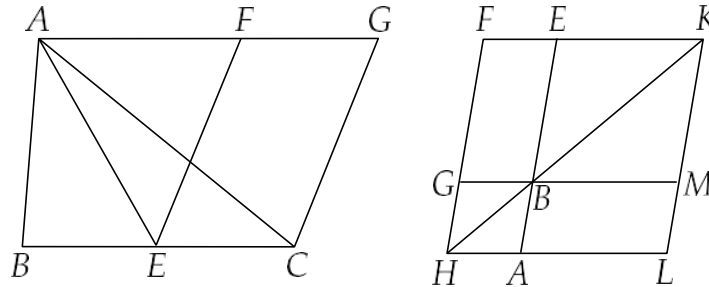


Figure 3: *Elements*, Propositions I.42, 44

I.42 provides a construction of a parallelogram equal to the given triangle $\triangle ABC$; in the accompanying diagram, E is the midpoint of BC , and $\angle CEF$ is chosen at will (see Fig. 3, left).

In I.44, Euclid shows the equality of the parallelograms $FEBG = BMLA$. While the first parallelogram is given, the second is constructed, with its side AB freely determined, whereas H, K, M, L are intersection points of the respective lines parallel to the sides FE or FG (see Fig. 3, right). The proof relies on CN3.

In the diagram accompanying Proposition I.45 (see Fig. 4, left), the figure ABCD represents a polygon divided into triangles, which are transformed into equiangular parallelograms (by I.42), and, by I.44, these parallelograms are converted into parallelograms with the same height. Then, by I.35, the parallelogram FLKM can be transformed into a rectangle.

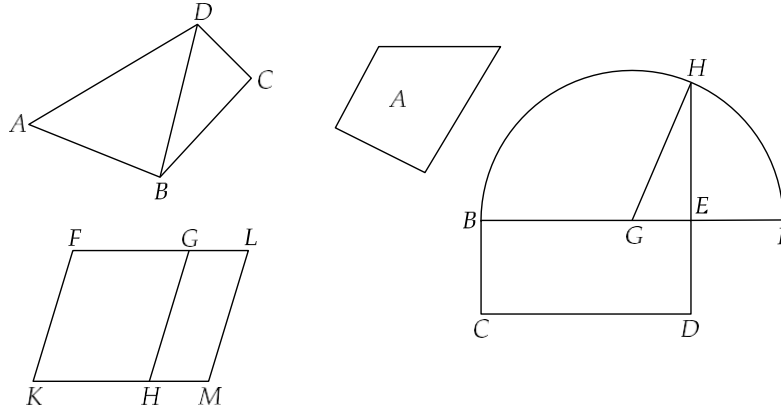


Figure 4: *Elements*, Propositions I.45 and II.14.

Finally, in Proposition II.14, a polygon, represented by the figure A, is turned into a rectangle, and Euclid shows that this rectangle is equal to the square on the line EH (see Fig. 4, right).

Hence, the equality of non-congruent polygons is effectively captured by the equality of their respective squares.

Regarding modern counterparts of that theory, Hilbert develops the theory of plane areas, and Hartshorne develops the theory of the content of figures.³

Note, however, that in Proposition I.48 Euclid assumes that the equality of squares entails the equality of their sides. To address this, Hartshorne introduces de Zolt's axiom, which is non-geometrical in character [Har00, p. 201]. Moreover, in order to manage the dissection component of Euclid's proof of I.47, as noted above, he implicitly invokes an axiom from ZFC set theory, namely, the existence of the empty set [Har00, p. 197].⁴

On the contrary, we can complement Euclid's theory on geometrical grounds by clarifying the equality of rectangles and defining an order relation as follows: $R_1 = R_2$ if and only if $EH_1 = EH_2$, where $R_1 = EH_1^2$ and $R_2 = EH_2^2$, and the squares EH_1^2 and EH_2^2 are constructed according to Proposition II.14. Respectively, we define $R_1 > R_2$ if $EH_1 > EH_2$.

On this basis, we can prove that the equality of squares entails the equality of their sides, as well as other propositions in Euclid's Books I–III, which Hartshorne proves by applying de Zolt's axiom

³ See [Hil99, Chapter 4], [Har00, § 22], and [Gio21] for a general account of theories of equal figures.

⁴ According to the definition of equal content [Har00, p. 197], one must assume the equality $Q = \emptyset = Q'$ to account for such dissections as BDL and CEL being equal to $BCED$.

[Har00, pp. 202–203]. Moreover, with these definitions, we can develop Euclid’s theory on a semi-Euclidean plane, whether Archimedean or non-Archimedean.

Mainstream mathematics recovers Euclid’s theory of equal figures by applying the formula for the area of a triangle $\frac{ah}{2}$, which, on the one hand, relies on the arithmetic of real numbers and, on the other, on the theory of similar figures [BP21, pp. 85–86].

In school mathematics, the formula ah for the area of a parallelogram – whence that for a triangle is derived – is typically justified by a dissection proof.

Finally, let us observe that, regarding the equality $sq\ ABFG = 2\triangle FBC$, in order to apply Proposition I.42, Euclid establishes that CA and AG form a straight line, since AG arises from a construction rather than as a prolongation of CA .⁵ Indeed, Proposition I.14 provides an appropriate argument. However, I.14 is the only proposition in Book I that Robin Hartshorne finds sufficiently unusual to require reinterpretation in order to “make sense” within Hilbert’s system.⁶

Ad 1. The justification of this equality occupies the bulk of the proof and constitutes its central and most detailed part. Euclid is particularly scrupulous in this regard. Notably, Proposition I.4 – in modern terms, the SAS congruence criterion for triangles – is exceptional. Its proof relies on the method of superposition and on the *ad hoc* principle that two straight lines cannot enclose an area.

Euclid applies superposition sparingly, essentially only in fundamental cases. The next use appears in Proposition I.8 – the SSS congruence criterion – and later in Proposition III.24, in connection with the rule *Three points determine a circle* [BP25].

Both issues – the superposition technique and the principle that a straight line cannot encompass a space – were not questioned in ancient times. From the modern perspective, these conceptual issues were resolved in Hilbert’s system by taking the SAS criterion as an axiom and by adopting as another axiom the rule: *Two points uniquely determine a straight line*.

Ad 0. In this section, we discuss the *κατασκευή* part, as the construction used in Proposition I.47 depends on the Parallel Postulate (Euclid’s Fifth Postulate). This naturally raises the question of whether there is any connection between the Pythagorean theorem and the Parallel Postulate

It follows from Saccheri’s theory that the very existence of a square, assumed in the construction, implies that all triangles are Euclidean. Given the Archimedean axiom, the Parallel Postulate follows [BP25, § 10].

⁵ The respective passage reads: “Two straight-lines (εὐθεῖαι) AC and AG , not lying on the same side, make the adjacent angles equal to two right-angles at the same point A on some straight-line BA . Thus, CA is straight-on to AG ,” while *straight line* (εὐθεῖα γραμμὴ) in this context means a line segment rather than an infinite straight line that separates the plane into two regions.

⁶ Hartshorne writes: “Rewrite the statement I.14 so that it makes sense in the Hilbert plane [...]” [Har00, p. 103].

Moreover, if we assume that Proposition I.47 can be formulated within the arithmetic of line segments or in a metric space, then the following holds: *If for every right-angled triangle with legs a , b , and hypotenuse c the identity $a^2 + b^2 = c^2$ is satisfied, then a Euclidean right-angled triangle exists.*⁷ Hence, Proposition I.47 implies the Parallel Postulate, given the Archimedean axiom.

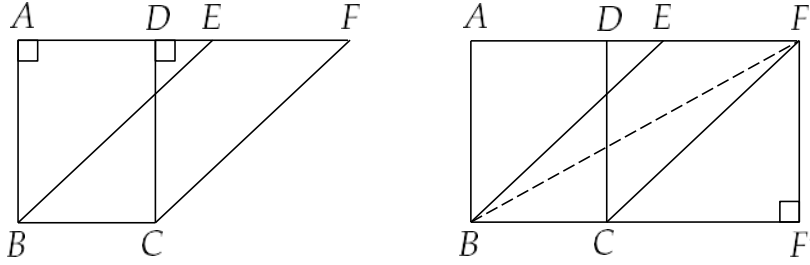


Figure 5: Revised Propositions I.35 (left) and I.36 (right).

Now, let us drop the Archimedean axiom and, instead of the Parallel Postulate, assume the existence of a square. That is, we proceed within a theory consisting of Euclid's Propositions I.1–27 together with the axiom There exists a square in place of Euclid's Fifth Postulate.

Then, by Saccheri's theory, the existence of a square implies that all triangles are Euclidean.⁸ Regarding steps (2) and (3) of Euclid's proof, we can proceed as follows.

Let ADBC be a rectangle and let EF = BC (see Fig. 5, left). The right-angled triangles $\triangle BAE$ and $\triangle CDF$ are equal, and by the same reasoning as in Proposition I.35, we conclude that the rectangle ADBC is equal to the figure EFCB.

By drawing from F a perpendicular to the ray $BC^{\rightarrow}$, we obtain the point F' (see Fig. 5, right). Since all triangles are assumed to be Euclidean, DFCF' is indeed a rectangle, and the triangles $\triangle BCF$ and $\triangle BEF$ are equal. Hence, $EFCB = 2 \triangle BCF$, and $ADBC = 2 \triangle BCF$.

Next, instead of drawing through the vertex A a line parallel to BD – as Euclid does in his proof – we draw from A a perpendicular to DE. The quadrilateral BDL has three right angles, and it follows that BDL is indeed a rectangle (see Fig. 6, left). Similarly, we can show that $BDL = 2 \triangle BAD$ (see Fig. 6, right).

⁷ To this end, consider an isosceles right-angled triangle $\triangle ABC$. Draw a perpendicular from B to the hypotenuse AC and show that it divides $\triangle ABC$ into two congruent isosceles right-angled triangles

⁸ Saccheri's original theory implicitly applies the Archimedean axiom to obtain this result; nevertheless, Hartshorne, by studying Saccheri quadrilaterals, reaches the same conclusion without assuming the Archimedean axiom [Har00, § 34].

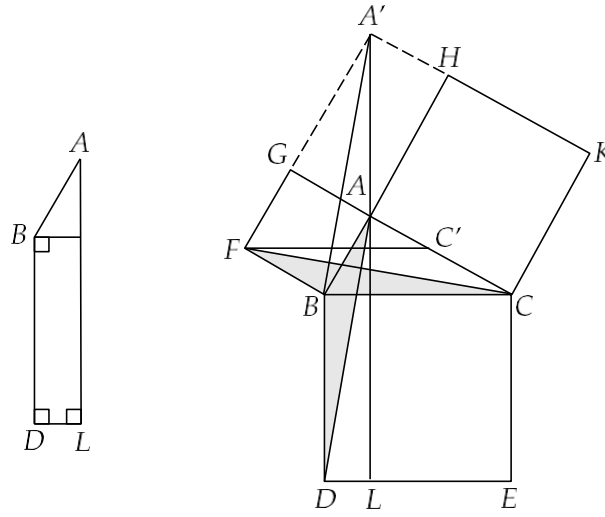


Figure 6: Revised Propositions I.47.

Now, the revised proof of I.47, is as follows:

$$\begin{aligned}
 \triangle FBC &= \triangle ABD \\
 FBC'C &= 2\triangle FBC = 2\triangle ABD = BDA'A \\
 sq\ ABFG &= 2\triangle FBC \\
 rec\ BDL &= 2\triangle ABD \\
 &\vdots \\
 sq\ ABFG, sq\ ACHK &= sq\ BDEC.
 \end{aligned}$$

The scheme is almost identical to that of the original proof; it differs only in the justification: instead of relying on the Parallel Postulate, we assume the existence of a square.

Finally, let us observe that, instead of the axiom *There exists a square*, we may adopt the axiom *There exists a Euclidean triangle*. A plane in which all triangles are Euclidean, yet the Parallel Postulate is not satisfied, is known as a semi-Euclidean plane. [BP21] provides a model of such a plane and includes a reference to a similar result by Max Dehn.

In summary, given the Archimedean axiom, the existence of a square assumed in the very formulation of the Pythagorean theorem implies the Parallel Postulate. However, the role of the Archimedean axiom in Euclid's geometry, as developed in Books I–IV, is largely a modern addition.⁹

We have shown that Euclid's proof can be reconstructed in a semi-Euclidean, non-Archimedean plane, which means that the crucial assumption concerns the existence of a Euclidean triangle rather than the Parallel Postulate.

3 Indian, Chinese, and Islamic Traditions

There are many so-called proofs by dissection of the Pythagorean theorem. Interestingly, they originate in the Indian, Chinese, and Islamic traditions, and later appear in early modern and modern European mathematics, rather than in ancient Greece. Today, they are often accompanied by an algebraic explanation, which diverges from their original geometrical sense.¹⁰ Therefore, in this section, we seek to focus on the original approaches.

Figure 7 (left) represents Bhāskara's (12th century) proof, also known as the *Behold* proof, owing to its one-word explanation. Bhāskara's remarks on this and other diagrams were indeed sparing in words; however, the very term *Behold* was attached to the diagram only in later commentaries [Kel05].

Figure 7 (right) represents Liu Hui's (3rd century) cut-and-paste proof. The red and blue diagram includes instructions – in Chinese characters – indicating how to remove and insert the pieces: *azure remove (b) – azure insert (b')*, *azure remove (a') – azure insert (a)*, *red remove (c) – red insert (c')* [Mar97, pp. 296–297].

Figure 7 (center) represents Thābit ibn Qurra's (9th century) version of the Pythagorean theorem. A Euclid-style proof begins with a construction: given a right-angled triangle $\triangle BAC$, draw a square on AB , set EF such that $EF = AC$, draw a square on EF , and draw DK such that $DK = AC$. The demonstration begins by establishing the equality of right-angled triangles:

$$\triangle BAC = \triangle CFG = \triangle GHK = \triangle KDB,$$

which implies that the corresponding sides of the quadrilateral BCGK are equal:

$$BC = CG = GK = KB.$$

⁹ For example, Lobachevsky identifies Euclidean geometry with the existence of Euclidean triangles and implicitly applies the Archimedean axiom [J9]. Klein, in presenting his model of hyperbolic geometry, claims that the sum of the angles in a triangle being π is equivalent to the Parallel Postulate [Kle71]; nowadays, we know that these are equivalent given the Archimedean axiom. Only Hilbert explicitly distinguishes between Archimedean and non-Archimedean geometries [Hil99]. For the history of the Archimedean axiom in the arithmetic of real numbers, see [BF20] – its role was recognized only in the 1880s.

¹⁰ See, for example, the algebraic account of Bhāskara's proof in [O'L10, p. 25], and an algebraic interpretation of Euclid's diagram II.7 in [Gre08, p. 28].

Next, an argument concerning the angles is given:

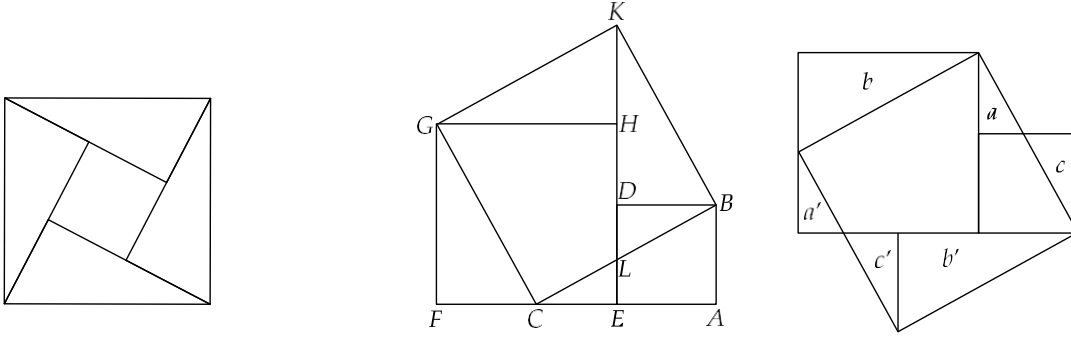


Figure 7: Proofs by dissection: Indian (left), Islamic (center), and Chinese (right).

- [1] The angles $\angle DBK$ and $\angle ABC$ are equal; hence, if we add $\angle DBC$ to each, the angle $\angle KBC$ equals $\angle ABD$ and is therefore a right angle. Similarly, the angle $\angle CGK$ is right.
- [2] Consequently, $BCGK$ forms a square, i.e., the square on BC .

Finally, a cut-and-paste type argument is presented:

Now the sum of the quadrilateral $GCLH$ and the triangle LDB , together with two of the equal triangles, forms the squares on AB and AC , and together with the other two, forms the square on BC [Hea56, vol. I, pp. 364–365].

Ad 1. This inference is based on the assumption $\angle DBA = \frac{\pi}{2}$, that is, the existence of the square $DBEA$.

Ad 2. This step assumes that the right-angled triangle $\triangle BAC$ is Euclidean, that is $\angle ACB + \angle CBA = \frac{\pi}{2}$

Indeed, the existence of a square, together with the assumption that the right-angled triangles used in the cut-and-paste technique are Euclidean, forms the basis of proofs by dissection. According to the discussion in the previous section, such arguments could also be carried out in a semi-Euclidean plane.

There is yet another assumption underlying dissection proofs: the concept of equal figures. In modern reinterpretations, this is not immediately evident, since algebraic approaches rely on the notion of area represented by an algebraic formula. However, the original proofs use an alternative to Euclid's concept of equal figures: figures F_1 and F_2 are equal if they can be divided into finitely many non-overlapping triangles $T_1^1, \dots, T_1^n$ and $T_2^1, \dots, T_2^n$, such that the corresponding triangles are pairwise congruent, that is, $T_1^i \cong T_2^i$ for each i .

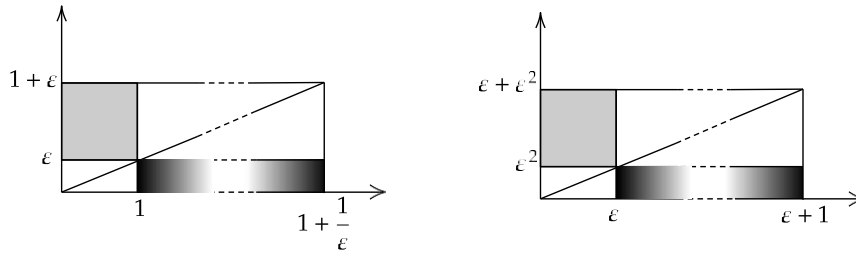


Figure 8: Non-decomposable rectangles in the Euclidean plane (left) and the semi-Euclidean plane (right).

To show that Euclid's theory is more general, let us consider a Cartesian plane $\mathbb{R}^* \times \mathbb{R}^*$, where $\mathbb{R}^*$ denotes the field of hyperreals.¹¹ Figure 8 (left) represents rectangles that, by Proposition I.44 (see Fig. 3), are equal.

In the diagram, ε denotes an infinitesimal; accordingly, $1/\varepsilon$ is infinitely large. The rectangle below the diagonal has a pair of infinitely long sides, with endpoints $(1, 0)$, $(1 + \frac{1}{\varepsilon}, 0)$ and $(1, \varepsilon)$, $(1 + \frac{1}{\varepsilon}, \varepsilon)$. Any finite set of triangles obtained by dissecting the rectangle above the diagonal (which is in fact a square) cannot cover these infinitely long sides. Hence, the two rectangles are not equal by dissection.

Figure 8 (left) is based on the model presented by Hilbert [Hil71, p. 63], which we have adapted to the Euclidean context. It can be further adapted to our model of the semi-Euclidean plane [BP21] as follows. The rectangles represented in Figure 8 (right) are, by Proposition I.44, equal. However, any triangle resulting from a dissection of the square above the diagonal has infinitesimally small sides, and no finite collection of such triangles can cover the unit-length sides of the rectangle below the diagonal.

Thus, even on the semi-Euclidean plane, Euclid's theory of equal figures is more general than equality by dissection.

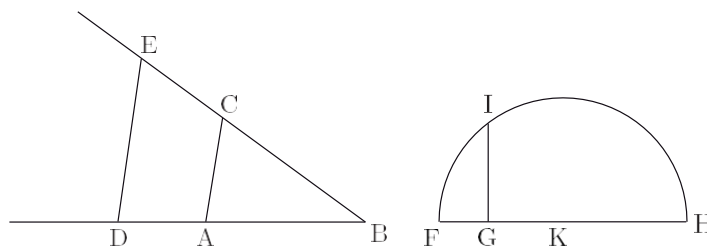
4 Descartes, *La Géométrie* (1637)

Christopher Clavius's edition of the *Elements* dominated the 17th century and was widely studied in Jesuit schools. His account of Proposition I.47 is essentially the same as Euclid's [Cla89, p. 229], so we may assume that Descartes learned this version at La Flèche. However, the reinterpretation he offers in *La Géométrie* is entirely new – one might even call it revolutionary. It first appears in his discussion of methods for finding the roots of quadratic equations. To situate this development, we begin with a brief introduction to his arithmetic of line segments.

4.1 Arithmetic of line segments

On the very first pages of *La Géométrie*, Descartes introduces new operations on line segments – product, division, and square root. The diagrams in Fig. 9 imitate the familiar Euclidean constructions of Propositions VI.12 and II.14; however, Descartes' aim is to assign to them a new meaning rather than to provide new proofs of the respective propositions.

¹¹ A Cartesian plane over an ordered field closed under the square root operation is a model of the Euclidean plane [Har00, pp. 144–145]

Figure 9: *La Géométrie*, p. 287

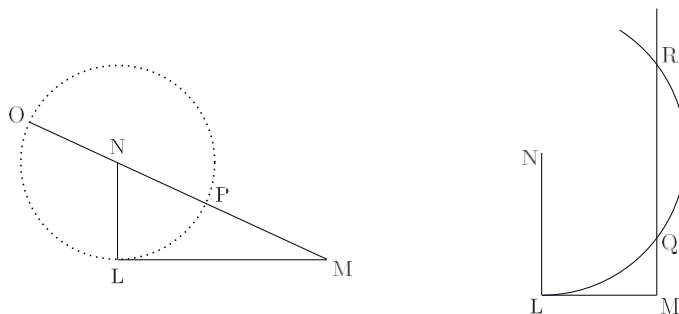
Thus, he introduces the concept of a unit line segment – interestingly, in his diagrams it is represented by unequal segments, such as AB in Fig. 9, left, and FG in Fig. 9, right – and declares BE to be the product of BD and BC , that is, $BE = BD \cdot BC$. He then takes BC to be the quotient of BE by BD , and GI to be the square root of GH . Moreover, when addressing specific problems, he rewrites the proportionality obtained from similar triangles, such as $a : b :: c : d$, in terms of his arithmetic of line segments, for example as $\frac{a}{b} = \frac{c}{d}$ or $b \cdot \frac{c}{d}$.

As a result, starting from Euclid's geometry, he derives modern algebraic formulas, which he then processes according to the rules of an ordered field – indeed, a real closed field [Bła24].

4.2 Solving a quadratic equation

In Book I of *La Géométrie*, Descartes shows how to construct, with straightedge and compass, the roots of a quadratic equation. These roots play a crucial role in dealing with Pappus' problem for four lines [Des37, pp. 325–334]. Later, in Book III, he shows how to use the parabola to solve cubic equations, and other curves to construct solutions to higher-order equations.

In what follows, we focus on quadratics, as these solutions unveil Descartes' novel reading of Proposition I.47 and several other fundamental Euclidean propositions. We stress that these are merely reinterpretations: Descartes does not aim to correct errors, clarify imperfections, or remedy supposed gaps in Euclid's text.

Figure 10: *La Géométrie*, pp. 302, 303

Thus, Descartes explains how to find the roots of the following equations:

$$\begin{aligned} (1) \quad z^2 &= az + b^2, \\ (2) \quad z^2 &= -az + b^2, \\ (3) \quad z^2 &= az - b^2. \end{aligned}$$

Ad 1. He writes (see Fig. 10, left):

Then this root (*racine*) or unknown line can easily be found. For example, if I have $z^2 = az + bb$, I construct a right triangle NLM with one side LM , equal to b , the square root of the known quantity bb , and the other side LN , equal to $\frac{1}{2}a$, that is, the half of the other known quantity which was multiplied by z , which I supposed to be the unknown line. Then prolonging MN , the base of this triangle, to O , so that NO is equal to NL , and the whole line OM is the required line z . This is expressed in the following way:¹²

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}. \text{ [Des37, 302 – 303].}$$

Descartes does not explain how equation (1) is related to the diagram Fig. 9 (left). In fact, it is a reinterpretation in his arithmetic of line segments of Euclid's Proposition III.36, which — in Euclid's terms — states that the square on DB equals the rectangle $DC.DA$ (rectangle with sides DC and DA), given that BD is tangent to the circle, or $\angle DBF = \frac{\pi}{2}$; see Fig. 11.¹³

Since Descartes takes $NL = \frac{1}{2}a$, $LM = b$, and $OM = z$, then III.36 in his arithmetic gives

$$OM \cdot PM = LM^2,$$

or equivalently,

$$z(z - a) = b^2,$$

which yields equation (1)

Indeed, from Descartes' perspective, equation (1) is already made evident by Fig. 10, left, and therefore requires no further explanation. Similarly, the solution

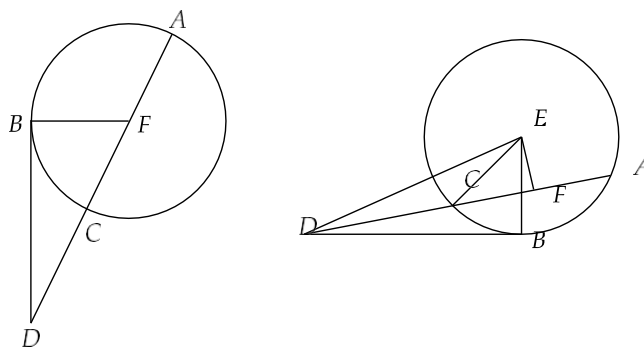
$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2} \quad (1)$$

needs no additional comment, as it follows directly from the same diagram.¹⁴

¹² Translations of *La Géométrie* follow [SL07], which also includes a reprint of the 1637 edition.

¹³ For a detailed discussion of Euclid's III.36, see [BP25].

¹⁴ We could check it *par le calcul*, in the same way Descartes verifies that certain lines are roots of the equation $z^3 = 3z - q$ [Des37, p. 397]. However, it is worth emphasizing that Descartes does not feel obliged to perform this check at this stage; see [Bla24].

Figure 11: *Elements*, III.36.

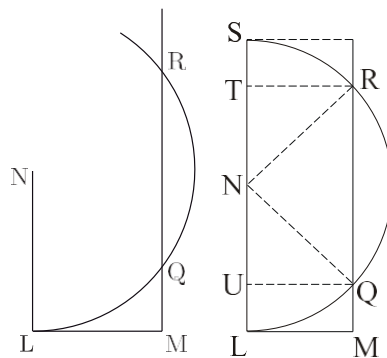
For the first time in history, the hypotenuse of a right-angled triangle is represented by an algebraic formula in the arithmetic of line segments. In Greek and early modern mathematics, the hypotenuse was considered merely as the side of a square equal to the squares on the legs of the triangle. In *La Géométrie*, by contrast, it becomes a line determined by the legs themselves – a right-angled triangle is treated purely in terms of its sides.

Ad 2. The solution for the second case does not introduce any new concept. Euclid's III.36 warrants the equation $y(y + a) = bb$, given that $y = PM$, $y + a = OM$, and $b = LM$.

But if I have $y^2 = -ay + bb$ here y is the quantity whose value is desired, I construct the same right triangle NLM , and on the hypotenuse MN lay off NP equal to NL , and the remainder PM is y , the desired root. Thus I have

$$y = \frac{1}{2} a + \sqrt{\frac{1}{4} aa + bb}. \text{ [Des37, 303]}$$

Similarly, the solution follows from the interpretation of Fig. 10.

Figure 12: *La Géométrie*, p. 303.

Ad 3. In the third case, Descartes refers to the second of Euclid's diagrams (Fig. 11, right), assuming that DB is shorter than the radius of the circle and that the line DA , which cuts the circle,

is perpendicular to DB . Given the assumptions $NL = \frac{1}{2}a$, $LM = b$, and $QM = z$, Proposition III.36 can be written in the form

$$z(a - z) = b^2,$$

which yields equation (3). The solutions are obtained by taking $z = NL + NT$ or $z = NL - NU$.

Finally, if I have $z^2 = az - bb$, I make $NL = \frac{1}{2}a$ and $LM = b$ as before; then, instead of joining the points M and N , I draw MOR parallel to LN , and with N as a center describe a circle through L cutting MQR in the points Q and R ; then z , the line sought, is either MQ or MR , for in this case it can be expressed in two ways, namely:

$$z = \frac{1}{2}a + \sqrt{\frac{1}{4}a^2 - b^2}$$

and

$$z = \frac{1}{2}a - \sqrt{\frac{1}{4}a^2 - b^2} \quad [\text{Des37, 303}]$$

In these solutions, the formula

$$\sqrt{\frac{1}{4}a^2 - b^2} \tag{2}$$

represents the leg NT of the right-angled triangle $\triangle NTR$ or the leg NU of the right-angled triangle $\triangle NUQ$. For the first time in history, a leg of a right-angled triangle is expressed by an algebraic formula within the arithmetic of line segments.

But there is even more.

4.3 Descartes' reinterpretation of I.47 and Euclid's gnomons

In Greek mathematics, the equality arising from the Pythagorean theorem (see Fig. 1),

$$AC^2 + AB^2 = BC^2,$$

where AC^2 abbreviates the term “square on AC ,” was never transformed to determine a leg of a triangle, for example,

$$AC^2 = BC^2 - AB^2.$$

To achieve equivalent results, the Greeks used figures called gnomons, which can be seen as the difference of two squares; in Fig. 13, *gnomon* $GDF = GF^2 - \text{grey square}$.

Its clearest example appears in Proposition II.14 (see Fig. 13, right), where the square on EH is proved to be equal to the rectangle contained by BE and EF ; below, we represent this rectangle as $BE.EF$.

The proposition is rather abstract, as neither the square EH^2 nor the rectangle $BE.EF$ is represented in the original diagram (see Fig. 4, right).

Here is the crucial part of proof II.14 in a schematized form (see Fig. 13):

$$\begin{array}{rcl}
 & \rightarrow & BE.EF, GE^2 = GF^2 \\
 GF = GH & \text{II. 5} & \\
 & \rightarrow & BE.EF, GE^2 = GH^2 \\
 \angle E = \pi/2 & \rightarrow & GH^2 = GE^2, EH^2 \\
 & \text{I. 47} & \\
 & \rightarrow & BE.EF, GE^2 = GE^2, EH^2 \\
 & \rightarrow & BE.EF = EH^2 \\
 & \text{CN3} &
 \end{array}$$

The conclusion, thus, is obtained by cancellation rather than subtraction.

On the other hand, a post-Cartesian account would run as follows. Let $GH = c$, $GE = a$, and $EH = b$. The area of the rectangle $CBED$ is $(c + a)(c - a)$. By the Pythagorean theorem, $c^2 - a^2 = b^2$. Hence,

$$(c + a)(c - a) = c^2 - a^2 = b^2.$$

However, neither $(c + a)(c - a) = c^2 - a^2$ nor $c^2 - a^2 = b^2$ finds a counterpart in Euclid's proof.

In algebraic stylization, Euclid's proof would be closer to the following:

$$\begin{aligned}
 (c + a)(c - a) + a^2 &= c^2, \\
 b^2 + a^2 &= c^2, \\
 (c + a)(c - a) + a^2 &= b^2 + a^2, \\
 (c + a)(c - a) &= b^2.
 \end{aligned}$$

A gnomon in Euclid’s proof underlies the reference to Proposition II.5. In this proposition, Euclid effectively asserts that the gnomon GDF , together with the grey square in Fig. 13, “is the whole square” (ὅλον ἐστὶ τὸ τετράγωνον) on GF .¹⁵

Gnomon GDF , which we prefer to view as the difference of two squares, $GF^2 - \text{grey square}$, plays in Euclid’s account the role of a figure in the dissection of a square: the gnomon GDF , together with the grey square, is the square GF^2 .

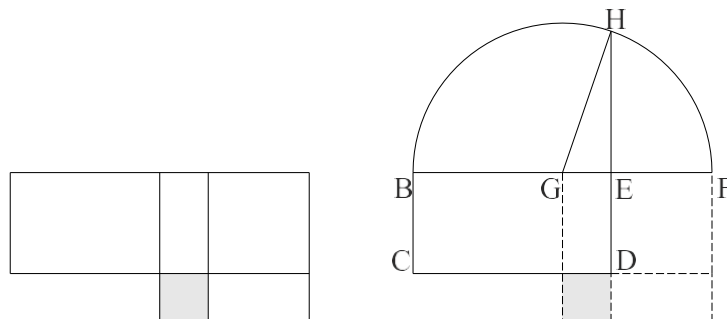


Figure 13: The scheme of application of Proposition II.5 in II.14.

Going back to the question of whether a form of dissection argument is present in the *Elements*, we claim that it appears only as an identity: for example, the gnomon GDF together with the grey square *is the square on* GF (Proposition II.4), and the rectangle BDL together with the rectangle CEL *is the square* $BDEC$ (Proposition I.47).

4.4 Descartes’ paradigm shift

Descartes’ phrase *square of*¹⁶ – which makes sense within the framework of *La Géométrie* – replaced *square on*, which refers back to the Euclidean paradigm. Since *La Géométrie*, a right-angled triangle has been treated in terms of line segments instead of squares as figures. By the end of the 19th century, this approach developed further as line segments were replaced by their lengths, that is, real numbers.

¹⁵ The names of the points in II.5 and II.14 differ; therefore, we do not cite the entire original phrase.

¹⁶ See, e.g., ss, *qui est le carré de la base, qui sont les carrés des deux* [Des37, p. 342]. For more textual evidence, see [Bla24].

We summarize this change in the foundational framework for the Pythagorean theorem as follows:

$$\square, \square = \square$$

$$a^2 + b^2 = c^2$$

$$a^2 + b^2 = c^2$$

Euclid's theory of equal figures
 Descartes' arithmetic of line segments
 Arithmetic of real numbers

5 Euler, *Introductio in analysin infinitorum* (1748)

Euler's reinterpretation of the Pythagorean theorem in terms of trigonometric functions concerns his novel approach to sin, cos, and tan in the context of the relationship between an arc and the corresponding chord of a given circle. To highlight this novelty, we first sketch his predecessors in that pursuit: Ptolemy's table of chords and Newton's series for the sine.

5.1 Numerical interpretations of the Pythagorean theorem: Ptolemy and Heron

Ptolemy identified the sine with the chord of a circle considered with respect to the arc it subtends and the corresponding central angle; see the chord AB , the arc AB , and the angle AOB in Fig. 5.1, left. In the *Almagest*, he relates arcs and chords by the formula

$$\text{arc} : 360 :: \text{chord} : 120. \quad (3)$$



Figure 14: Left: Ptolemy, *Almagest*, p. 57 (the centre O and the dotted line added). Right: a reinterpretation of the Pythagorean theorem.

When an arc corresponding to a central angle α is such-and-such a part of the circle, divided into 360 parts, the chord $Crd \alpha$ is such-and-such a part of the diameter, divided into 120 parts. Yet there is no explicit formula relating the unit of the circle to the unit of the diameter – these are magnitudes of different kinds and cannot be compared. Nevertheless, formula (3) enables one to express a proportion ($\acute{\alpha}\nu\alpha\lambda\omicron\gamma\iota\alpha$) of ratios involving the unit of the circle:

$$\text{arc of } 1^\circ : 360 :: Crd 1^\circ : 120$$

Ptolemy approximated the unit of the circle by approximating $\text{Crd } 1^\circ$; in the Babylonian system, it is

$$\text{Crd } 1^\circ \approx 1;2,50.$$

In *Almagest*, $\text{Crd } \alpha$ is thus a number assigned to the line subtending the angle α . Hence, $\text{Crd } 60^\circ = 60$, since it is the radius of a circle with diameter 120, or equivalently, the side of hexagon inscribed in the circle. Likewise, $\text{Crd } 90^\circ \approx 84;51,10$, as it is an approximation of $\sqrt{60^2 + 60^2}$ – the hypotenuse of an isosceles right-angled triangle with sides 60 and 60, or equivalently, the side of a square inscribed in the circle. This second chord is clearly a numerical reinterpretation of the Pythagorean theorem.

In a series of propositions providing geometric bases for the numerical determination of chords, Ptolemy applies the Pythagorean theorem to a triangle constructed on the diameter and arrives at the following equation (see Fig. 5.1, right):

$$[\text{Crd}(180^\circ - \alpha)]^2 + [\text{Crd } \alpha]^2 = 120^2. \quad (4)$$

It provides a general numerical reinterpretation of the Pythagorean theorem.

Ptolemy divided the circle into 720 parts, corresponding to increments of $1/2^\circ$. Using the chord formulas for $\text{Crd}(\alpha \pm \beta)$ and $\text{Crd}(\frac{\alpha}{2})$ – formulas he himself discovered – he was able to compute the chord for every arc from $1/2^\circ$ to 180° . In doing so, he effectively combined Euclid’s geometry with Babylonian arithmetic [Too84, Book I, §§ 10–11].

An earlier arithmetical reinterpretation of Euclid’s Proposition I.47 is found in Heron’s *Metrica*. In Proposition I.5, Heron considers a right-angled triangle with hypotenuse 13 and one leg 5, and states that the other leg is equal to 12. This second leg is the height of a triangle with sides 13, 14, and 15 [Sch03, pp. 12–16]. These considerations lead Heron to his famous formula, which allows one to determine the area of a triangle from the lengths of its sides – both area and length being expressed in terms of numbers of monads ($\mu\omicron\nu\acute{\alpha}\delta\epsilon\varsigma$); however, the monads measuring lines and those measuring areas are of different kinds and are therefore incomparable.¹⁷

5.2 Series for *arcsine* and *sine*: Newton

In *Analysis by Equations of an Infinite Number of Terms* [New45], Newton applies his technique of infinite series. In Fig. 15 (right), the line TD is the tangent to the semicircle $ADLE$, and the chord AD (not represented in the diagram) is the sine of the angle ACD . Newton seeks to determine the arc AD in terms of the line AB , where $AB = x$. To this end, he introduces the “indefinitely small rectangle” $BGBK$, and sets $AC = \frac{1}{2}$. The point H lies on TD and, at the same time, is the vertex of the triangle HDG and lies on the semicircle.

¹⁷ For the source text, see [Sch03, pp. 12–24], and for a commentary, see [B-la24].

Following Descartes' interpretation of the Pythagorean theorem, as represented in (2), Newton finds that

$$BD = \sqrt{\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2} - x\right)^2} = \sqrt{x - x^2}. \quad (5)$$

From the similarity of triangles $\triangle TDB$, $\triangle HDG$, and $\triangle CDB$, he derives the proportions

$$KB : HD :: BT : DT, \quad BT : DT :: BD : DC,$$

and obtains the following ratios:

The ratio $HD : BK$ represents *the moment of the arc AD* to the *moment of the base AB*. The proportion

$$HD : BK :: DC : BD$$

is turned into the equality

$$\frac{HD}{BK} = \frac{1/2}{\sqrt{x - x^2}}.$$

Assuming $BK = 1$, he obtains¹⁸

$$HD = \frac{1}{2} \sqrt{x - x^2}.$$

By the binomial theorem, Newton expands HD into a series:

$$\frac{1}{2} \sqrt{x - x^2} = \frac{1}{2} x^{1/2} - \frac{1}{4} x^{3/2} + \frac{3}{16} x^{5/2} - \frac{5}{32} x^{7/2} + \frac{63}{512} x^{9/2} - \dots$$

Via term-wise integration, he obtains that “the length of the arc is”

$$x^{1/2} \left(1 + \frac{1}{6} x + \frac{3}{40} x^2 + \frac{5}{112} x^3 + \frac{35}{1152} x^4 + \frac{63}{2816} x^5 + \dots \right).$$

¹⁸ Because of this assumption, there are two unit segments throughout the argument: the infinitesimal BK and, say, the standard AE . Nevertheless, this does not lead to a contradiction, since Newton compares only a ratio of infinitesimals with a ratio of finite segments.

Then Newton continues:

After the same manner, by supposing CB to be x , the radius CA to be 1, you will find the arc LD to be

$$x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots \quad [\text{New45,337}]. \quad (1)$$

That is, given $BC = \sin \angle DCL = x$ and $CA = 1$, the arc DL equals the series (6). In modern notation:

$$\arcsin x = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots$$

In the next section, Newton seeks to determine the sine in terms of its arc. In Fig. 15, right, the line AB represents the sine of the angle $\angle \alpha AD$ (line DA is not represented in the diagram), and αD denotes the arc. Given $\alpha D = z$, $AB = x$, and $A\alpha = 1$, it follows from (6) that:

$$z = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{5}{112}x^7 + \dots$$

By his ingenious technique of finding inverse series, Newton determines the relation between z and x as follows:

$$x = z - \frac{1}{6}z^3 + \frac{1}{120}z^5 - \frac{1}{5040}z^7 + \dots$$

In modern terms, this is the series for sine, namely:

$$AB = \sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \frac{1}{7!}z^7 + \dots$$

Thus, the series for $\arcsin z$ and $\sin z$ enable us to approximate the arc subtending an angle of 1° and the chord $Crd 1^\circ$.

The crucial assumption in Newton's derivation is the ratio of the infinitesimal line segments KB and HD (see Fig. 15, left), whereas the arc $\alpha D = z$ used in expanding the series for the sine appears to be standard (see Fig. 15, right).¹⁹

Euler begins his analysis with the diagram in Fig. 15 (right). He assumes that z – corresponding to Newton's notation $\alpha D = z$ – is an infinitesimal arc of the unit circle, and he takes βD to represent the sine of the angle $\angle \alpha AD$.

¹⁹ For a detailed analysis of Newton's approach, see [B-l25].

Although sine and cosine are introduced as line segments, Euler understands the tangent not merely as a line segment but also as the ratio of the sine to the cosine. In this way, the Euclidean notion of ratio is treated explicitly by Euler as a numerical quantity.

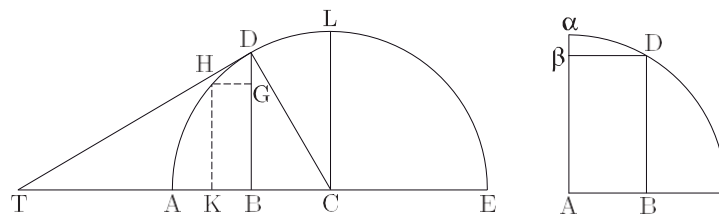


Figure 15: Newton, *Analysis of Equations*, p. 336 (left) and p. 338 (right).

5.3 Series for *sin* and *cos*. Euler

Euler's *Introductio in analysin infinitorum* [Eul48] is considered the beginning of modern trigonometry. Here is the founding supposition of its Chapter VIII (§§ 126–142):

We let the radius, or total sine, of a circle be equal to 1 (*sinus total*=1); then it is clear enough that the circumference of the circle cannot be expressed exactly as a rational number. An approximation of half of the circumference of this circle is

$$3.14159265358979323846264338327950288419716939937510820974944592307 \\ 8164062862089986280348253421170679821480865132723066470938446 \dots$$

For the sake of brevity, we will use the symbol π for this number. We say, then, that half of the circumference of a unit circle is π , or that the length of an arc of 180° is π . [Eul88, p. 101]

The phrase “total sine” refers to the radius of the circle.

Euler's account is novel primarily because the unit circle establishes a new standard. Indeed, for Ptolemy and Newton, the sine was a specific line segment; referring to Fig. 16, it was PB for Ptolemy and PA for Newton. Euler, however, explicitly shows that his sine can take negative values. Therefore, it should be regarded both as a line segment and as the coordinate of a point rotating on the unit circle.

We always assume that the radius of the circle is 1 and let z be an arc of this circle. We are especially interested in the sine and cosine of this arc z . Henceforth, we will signify the sine of the arc z by $\sin z$, and for the cosine of the arc z we will write $\cos z$.

Since π is an arc of 180° ,

$$\sin 0 = 0, \quad \cos 0 = 1, \quad \sin \frac{\pi}{2} = 1, \quad \cos \frac{\pi}{2} = 0,$$

$$\sin \pi = 0, \quad \cos \pi = -1, \quad \sin \frac{3\pi}{2} = -1, \quad \cos \frac{3\pi}{2} = 0, \quad \sin 2\pi = 0, \quad \cos 2\pi = 1.$$

Every sine and cosine lies between $+1$ and -1 . Further, we have

$$\cos z = \sin\left(\frac{\pi}{2} - z\right), \quad \sin z = \cos\left(\frac{\pi}{2} - z\right), \quad \sin^2 z + \cos^2 z = 1.$$

Besides these notations, we also mention that $\tan z$ indicates the tangent of the arc z , and $\cot z$ the cotangent of the arc z . We agree that

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z} = \frac{1}{\tan z},$$

all of which is known from trigonometry (Euler 88, p.102).

Based on further developments, we may assume the following (see Fig. 16):²⁰

$$\sin z = PA, \quad \cos z = OA, \quad \tan z = \frac{\sin z}{\cos z} = CB, \quad \text{arc } PB = z.$$

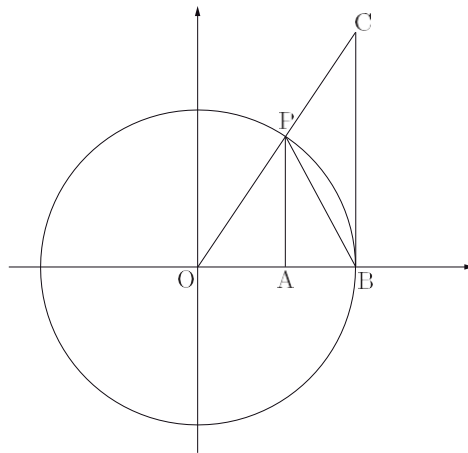


Figure 16: Diagram illustrating Euler's definitions of trigonometric functions.

²⁰ To be clear, Fig. 15 is not included in (Euler 1748).

Yet the central identity, from our perspective, is explicitly given as

$$\sin^2 z + \cos^2 z = 1. \quad (7)$$

That is Euler's trigonometric reinterpretation of the Pythagorean theorem.

In the next step, he factorizes it as:

$$(\sin z + i \cos z)(\sin z - i \cos z) = 1$$

and develops it further. Interestingly, z in this formula represents an infinitely small arc of the unit circle:

Let the arc z be infinitely small, then $\sin z = z$ and $\cos z = 1$. If n is an infinitely large number, so that nz is a finite number, say $nz = v$, then, since $\sin z = z = v/n$, we have

$$\begin{aligned} \cos v &= 1 - \frac{v^2}{1 \cdot 2} + \frac{v^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{v^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \dots \\ \sin v &= v - \frac{v^3}{1 \cdot 2 \cdot 3} + \frac{v^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \frac{v^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots \end{aligned}$$

It follows that if v is a given arc, by means of these series, the sine and cosine can be found [Eul88, p. 107].

For a detailed analysis of Euler's development, see [BP23].

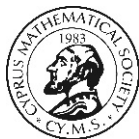
In modern mathematics, the trigonometric reinterpretation of the Pythagorean theorem takes nearly the same form as Euler's; the difference is that z represents the radian measure of an angle $\angle POA$. For a discussion of the implicit assumptions underlying the identification of an arc of a circle with the radian measure of the corresponding angle, see [Bła25].

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*Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026*

The Role of Women in the Pythagorean School and the Contribution of Themistoclea:

Mathematical, Philosophical, Ethical, and Spiritual Influences

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ABSTRACT

This paper explores the historical participation of women within the Pythagorean School, challenging the modern misconception that female involvement in STEM and philosophy is a strictly contemporary phenomenon. By examining the life and influence of Themistoclea, a Delphic priestess and teacher to Pythagoras, the research highlights how the Pythagorean tradition integrated spiritual, mathematical, and ethical wisdom. The study further discusses the contributions of figures such as Theano, Damo, and Myia, who served as leaders, guardians of manuscripts, and practitioners of applied philosophy. Utilizing historical accounts from Iamblichus and Diogenes Laërtius, the article argues that the Pythagorean School established an early model of gender equality and intellectual collaboration. These ancient precedents provide a valuable framework for contemporary educational policies aimed at increasing female representation in Science, Technology, Engineering, and Mathematics (STEM) by offering historical role models and reinforcing the ethical dimensions of scientific inquiry.

Keywords

Arithmosophy, Ethics, Mathematics, Philosophy, Pythagorean School, STEM, Themistoclea, Women in Science



First and foremost, it is important to clarify that this presentation is not an attempt to promote feminist approaches. Rather, it is a stimulus arising from contemporary reflection and research findings regarding the participation of women in **STEM** (Science, Technology, Engineering, Mathematics), which remains low compared to that of men. This trend is often attributed to gender stereotypes that discourage women from pursuing sciences.

However, modern examples of excellence debunk these prejudices. A characteristic case is that of the Cypriot scientist **Kyriaki Michailidou**, who managed to transcend stereotypes and overcome the social barriers that keep women away from STEM. She proved that mathematics can provide solutions to critical social problems. Notably, Dr. Kyriaki Michailidou was honored by UNESCO with the International Al Fozan Award for the Promotion of Young Scientists in STEM for her research titled: *"Mathematics as a Weapon Against Cancer."*

History teaches us that the involvement of women in mathematics and engineering is not a recent phenomenon but has deep roots in antiquity. The central goal of this work is to highlight how the **Pythagorean School**, under the guidance of Pythagoras, managed to overcome narrow-minded patriarchal perceptions. By examining the role of **Themistoclea** and other women, we attempt an overview of their contribution not only to the sciences but also to the ethical and spiritual values that govern Western thought.

Furthermore, this presentation fundamentally seeks to highlight the greatness of Pythagoras' personality. He succeeded in transcending narrow social views within a patriarchal society and stereotypes that still appear even today.

THEMISTOCLEA: The Teacher of the "Father of Philosophy and Mathematics"

Themistoclea, also known as Aristoclea, was a priestess at Delphi during the 6th century BCE. She demonstrated that philosophy was not limited to logical, written tradition but could draw from spiritual and oracular wisdom. According to sources such as **Aristoxenus** (4th century BC) and **Diogenes Laertius** (3rd century AC), Themistoclea taught Pythagoras Mathematics, the "ethical dogmas," and the principles of ethics.

Pythagoras greatly admired her knowledge and wisdom. According to Aristoxenus, his admiration and respect for this woman were the reasons he later accepted women into his School — as students and as teachers — among whom were his wife and daughters.



Themistoclea was renowned for her mathematical knowledge. Not only was she recognized by Pythagoras' contemporaries as his teacher in geometry and arithmosophy (number wisdom), but Pythagoras himself cited her as his mentor. Such was her love for geometry that it is said she decorated the Temple of Apollo with geometric designs.



The temples of the Greek Gods, with their priests and priesthoods, did not function exclusively as religious centers. The famous and known in all the ancient and contemporary world, the **Oracle of God Apollo at Delphi** also functioned as a university. Themistoclea taught Mathematics... Geometry... but above all, she taught **Ethics**! She taught anyone who was willing to learn.

In this land, there were many who wished to learn, who were taught, and who later became teachers themselves — those who were initiated into occult knowledge and, in turn, initiated the next generation.



The lack of direct written works or biographical details for Themistoclea does not diminish her importance; rather, it suggests a different form of spiritual authority in antiquity. As **Pythia**, the divinely inspired priestess of the Delphic Oracle, her authority was based on spiritual wisdom and prophetic insight, rather than the rational, prose-based philosophical tradition of male philosophers like Plato or Aristotle. Her legacy challenges the traditional definition of philosophy, suggesting that non-rational and feminine forms of knowledge and authority should be considered legitimate sources of wisdom. Pythagoras' choice to learn from a priestess shows an acknowledgment that philosophy cannot be confined to a single, male, logical tradition, thus enriching Pythagorean thought with a spiritual dimension that transcends simple logic.

Prominent Women of the Pythagorean School

There is historical evidence that a series of other eminent women participated in the mathematical and philosophical life of Ancient Greece, and specifically the Pythagorean school. Names such as **Theano**, **Damo**, and **Myia** are attributed to women associated with the Pythagorean tradition through teaching, writing, or organizational action. They contributed to administration, the preservation of writings, and the application of philosophy to daily domestic life. Furthermore, texts attributed to women like **Perictione** and **Phintys** laid the foundations for extending philosophical thought beyond the traditional divide between the private and public spheres.

Theano: The Administrative and Productive Thinker

Theano is traditionally referred to as the wife of Pythagoras and the co-leader of the school after his death. Sources attribute to her the authorship of treatises on a wide range of subjects, including mathematics, physics, medicine, and child psychology. She is also credited with introducing the concept of the "**Golden Ratio**."



"Alegoría de la Geometría" (Allegory of Geometry), from Laurent de la Hyre

Damo: The Guardian of the Legacy

Damo, daughter of Pythagoras and Theano, played a crucial role in preserving the Pythagorean tradition. After her father's death, she was entrusted with the custody of his writings. Her devotion was so strong that she refused to sell the manuscripts even in extreme poverty, because Pythagoras had forbidden sharing his teachings with strangers.

Myia: The Ethicist of the Household

Myia, also a daughter of Pythagoras, is known for her "Letter to Phyllis," in which she provided advice on raising an infant. Her letter applied the Pythagorean principle of moderation to childcare, suggesting the avoidance of extreme behaviors. This contribution suggests that the Pythagorean philosophy was not restricted to abstract concepts, but penetrated everyday life and gave the 'oikos' a high moral and spiritual value.

Perictione and Phintys: The Theorists of Female Virtue

Beyond the historical references a set of pseudepigraphal treatises is attributed to women like Perictione and Phintys. Works such as *"On Harmony"* and *"On the Self-Control of Women"* reformulated core Pythagorean concepts, such as harmony and prudence, through a distinctly female perspective.

THE PYTHAGOREAN SCHOOL AND ARITHMOSOPHY

The Pythagoreans advocated that **Mathematics is the path to the liberation of the soul**. Based on Pythagoras' belief that "the elements of numbers are the elements of all beings," they assigned the highest importance to Arithmetic. They believed that a number is not something perceived through the senses, but through the intellect.

Indicative Dialogue (Narrative Excerpt):

Pythagoras: My brothers, in the search for truth, knowledge is not a possession, but a journey. As Themistoclea showed me the way, so I invite you to walk together.

Student: Master, why are numbers so important? Are they not just measurements?

Pythagoras: Not just measurements. Numbers are the soul of the world. **3** is harmony, **4** is justice, **10** is perfection. Everything that exists can be expressed in proportions.

Another Student: And music? How does it connect to mathematics?

Pythagoras (picking up a lyre): Listen. When the string is doubled, the sound changes in a predictable way. Harmony is mathematical. Music is number turned into sound.

Student (in awe): It is as if we see the world with a new pair of eyes!

SPECIFIC ROLES OF WOMEN IN THE SCHOOL

Women played a significant role in the spiritual and ethical legacy of the school and in Pythagoreanism as a whole.

Ancient sources, spanning from the Classical era to Byzantium, consistently highlight the prominent role of women in Pythagoreanism. A Pythagorean woman was not merely a follower; she was considered " **an intellectual, a thinker, a teacher and a philosopher.**" This fundamental perspective sets the stage for exploring the multifaceted roles these women played.

In Mathematics

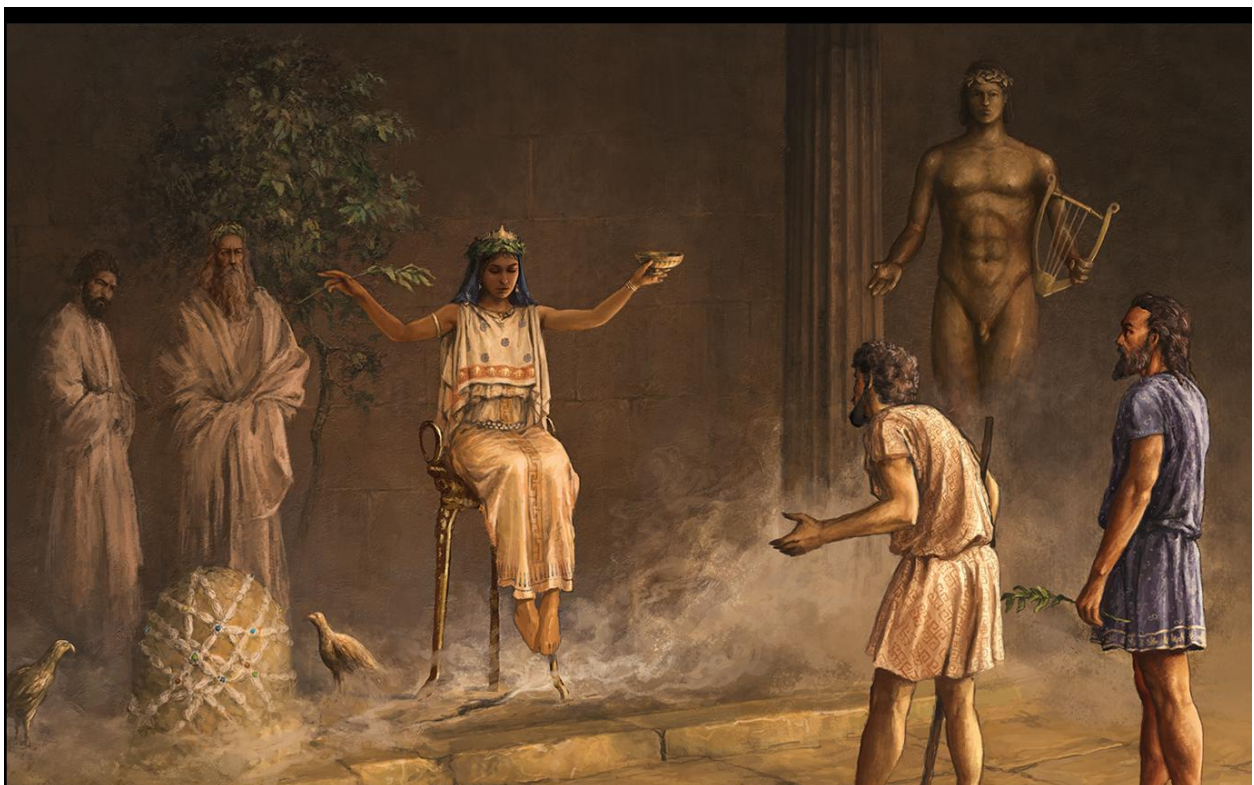
- **Theoretical Work and Teaching:** Women appear to have developed and taught mathematical theories concerning arithmetic, harmony, and proportions—elements central to Pythagorean mathematical thought.

- **Practical Application of Mathematical Knowledge:** Women participated in calculations, measurements, and the application of numerical principles in music, architecture and the technology of the era, acting as conduits for turning theory into solutions for practical problems.
- **Mediation of Mathematical Tradition:** Through teaching within the community and family pedagogy, women contributed to the transmission and preservation of mathematical knowledge.

In Natural Sciences and Natural Philosophy

- **Physiological and Cosmological Discussions:** Women appear in discussions regarding nature, the harmony of bodies and cosmological models linked to numerical ratios. The numerical interpretation of nature was a primary domain of Pythagorean work.
- **Experimentation and Technical Knowledge:** At the level of technical applications and measurements, women seem to have been involved in practical processes that we would today call "applied sciences"—specifically those related to musical acoustics and the measurement of harmony (the study of the numerical ratios of the musical scale).

Women are presented as active members of the Pythagorean tradition, engaged in the mathematical and physico-philosophical activities of the school. They served not only as "students" in a formal sense but also as **teachers, theorists, and exponents** of mathematical and theological discourse.



THE POSITION OF WOMEN IN THE COMMUNITY

The institutional legacy of Delphi was followed by Pythagoras in Croton, where he organized the **Omakhoion** (the common school) according to principles that promoted gender equality and the participation of all in spiritual, mathematical, and ethical studies. Women were not restricted to decorative roles: they participated in research, teaching, and initiatory rites, establishing a tradition of collaboration that persisted in later philosophical schools.

The Pythagorean Brotherhood

The Pythagorean Brotherhood, a philosophical and religious community founded by Pythagoras around the 6th century BC, was a secret society that combined elements of religion, philosophy, and political action. Within this framework, the inclusion of women was not merely an incidental aspect, but a **radical act** that distinguished it from the prevailing social norms of the ancient Greek world.

Sources explicitly state that the Pythagorean community included both men and women in a state of "**perfect equality**" based on spiritual merit rather than gender. Iamblichus provides a list of 235 famous Pythagoreans, seventeen of whom are women, acknowledging their active participation. The Pythagorean approach intellectually reframed existing gender roles, granting feminine virtues philosophical and even proto-political significance.

Improving the Quality of Life

Pythagoras was among the first to show understanding of the challenges women faced and enforced monogamy upon their husbands. For women, participation in the Pythagorean way of life represented a major improvement over the lifestyle imposed on most women of the era. For example:

- Pythagorean women were permitted to drink wine.
- They slept on linen sheets rather than wool, which was considered both more pleasant and more hygienic.
- Their husbands were required to respect them, refrain from abuse, and remain faithful.

The Common Ethical Code

Pythagoras urged a life that was "virtuous and pious." Pythagorean philosophy established a **common ethical code** for both men and women. The moral "regulations" of the *Golden Verses* were addressed to both genders and included virtues such as respect for the gods and parents, justice, tolerance, self-restraint, decency, modesty and wisdom.

Elevation of Feminine Virtues

Beyond the common code, Pythagorean ethics went beyond simply reinforcing the role of the "good wife." While it was noted that women had "certain additional duties suitable for their gender," this is interpreted as an **elevation of traditionally feminine roles into philosophical principles**. For women, "harmony" was identified with "prudence and temperance"—virtues considered fundamental to the soul.

Women were regarded as the "basic foundation of the household," and their behavior reflected upon the entire family, turning the home into a field of philosophical and ethical exercise. Elevating the household to a space of philosophical practice challenged the traditional Athenian dichotomy between the private, female-dominated *oikos* (home) and the public, male-dominated *polis* (city).

Conclusion: A Legacy Beyond the Home

The Pythagorean school stood as a rare and significant exception to the social norms of ancient Greece - a space where women could be intellectuals and philosophers. Their position was not merely symbolic; women were active members whose spiritual and ethical worth was fully recognized.

Implications for Modern Education and STEM

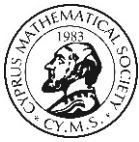
Female figures of knowledge and historical tradition can fuel modern initiatives to strengthen female participation in **STEM** through historical role models, educational content, and ethical support.



Pythagoras and Damo, detail from The School of Athens, 1509. Stanza della Segnatura, Vatican. Credit: profzucker, Flickr,

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Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026

Pythagoras: Yesterday, Today, Tomorrow

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ABSTRACT

Pythagoras is one of the most multifaceted figures of antiquity, a person who succeeded in bridging science, philosophy, religion, and art. This article, attempts to summarize the course and the significance of this remarkable personality throughout the centuries up to the present day and into the future. “Yesterday,” in order to recall his life and teachings, the environment in which he worked, and the foundations he laid for mathematics, science, and philosophy. “Today,” in order to examine how his ideas survived, transformed, and continued to inspire science, technology, and culture as a whole. And “Tomorrow,” in order to reflect on what Pythagorean thought may mean for the challenges of our own century, and for the need of a more integrated, more harmonious understanding of knowledge and education, with practical examples e.g., some STEAME applications. The well-known two-dimensional Pythagorean Theorem will be presented together with a philosophical approach to it, as well as its three-dimensional extension and its connections to the speed of light, to distances between planets and to music, and explore whether the ideas of Pythagoras still possess enduring mathematical and pedagogical value. The aim is to present Pythagoras both as a historical figure and as a source of inspiration that helps us reflect on the relationship between number and harmony, science and philosophy, knowledge and education - then, now, and in the future.

Keywords: *Pythagoras, Omakoeion, Tetraktys, EIII-STEAME, Philosophical Interpretation of Pythagorean Theorem, magic numbers 432 and 108, Pythagorean teachings guidance, School of the Future, the answer is always love.*

Pythagoras Yesterday

Pythagoras was born on Samos Island around 580 BCE (*), at a time when Greek thought was taking its first significant steps towards philosophy and science. As a young man, he travelled to places such as Egypt (where he studied with the priests), Babylon (where he studied with the Magi), and (possibly) India and other regions, significantly enriching his knowledge in mathematics, astronomy, religious traditions, and more. These influences shaped the distinctive system of thought that he would later develop.

Around 530 BCE, he settled in Croton, in Magna Graecia - the Greek colonies of Southern Italy. There he founded the *Omakoeion*, a school that was simultaneously a mathematical and philosophical centre and also a community of shared life, with strict rules, a common way of living, and a distinctive form of initiation. The *Omakoeion* was a unique institution, where mathematical research was combined with ethics and philosophy, offering an unparalleled model in antiquity. Everything which was taught was sacred (and thus secret). For example, the shape and the philosophical meaning of Tetraktys was one of the very top secrets.



Figure 1: Tetraktys

At the core of Pythagorean teaching stood the concept of number. Number was not merely a tool of measurement, but the very foundation of the order of the world. Through numbers, Pythagoras and his followers perceived harmony, proportion, and the discovery of a deeper balance governing the cosmos. The connection between number and music - how simple mathematical ratios produce pleasing sounds and melodies-served as proof that nature “speaks the language of numbers.”



Figure 2: An instrumental approach to Pythagoras' Monochord as a tool for highlighting musical intervals/discourses. Construction by the musician and instrumentalist Efxifios Satsias with measurements by the educator (philologist and ethnomusicologist) Christodoros Mnassonos

This teaching extended from mathematics and music to astronomy and philosophy. The planets were thought to move in harmonious ways, producing the so-called “*harmony of the spheres*,” an idea that would inspire philosophers and artists for centuries.

Among the mathematical achievements associated with Pythagoras, the Pythagorean Theorem naturally stands out. Although the relationship between the sides of a right triangle may have been known earlier in other cultures, the first systematic *proof* is attributed to Pythagoras and his school. The theorem is not merely a geometric tool; it is a demonstration of how logic and proof can transform empirical knowledge into science. There also exists the three-dimensional “Pythagorean Theorem,” which will be presented later both theoretically and practically.

Pythagoras’ work was not universally understood nor accepted in his own time. His ideas faced criticism, and his school encountered opposition, leading eventually to conflict. However, the seed had already been planted, and Pythagorean thought travelled across centuries, becoming an integral part of the philosophical and scientific tradition through his followers, through Plato, Aristotle, and later Neopythagoreans and Neoplatonists.

His intellectual lineage continued through figures such as Kerkops, Brontinus, Theano (his wife), Hippasus, his daughters Damo and Arignote, Alcmaeon, Iccus of Tarentum, Ameinias, Parmenides, Epicharmus, Empedocles, Philolaus, Lysis, Ocello, Cleinias, Archytas, Timaeus of Locri, Simmias of Thebes, Diotima (teacher of Socrates), Perictione (mother of Plato), Ecphantus of Syracuse, Leucippus, Democritus (who studied under Mnesarchus, Pythagoras’ second son), Plato, Aristotle (teacher of Alexander the Great), the students of Plato such as Eudoxus of Cnidus, Speusippus, Theaetetus, Xenocrates, Polemon, Xenophilus, and many others up through Cicero, Vitruvius, Pliny the Elder, Ammonius Saccas, Plotinus, Proclus, Cosimo de’ Medici, Marsilio Ficino, and beyond.

Summarizing the “yesterday,” Pythagoras bequeathed three fundamental elements:

1. The recognition of number as the foundation of reality
2. The idea of harmony as the bond between the parts of the world

3. The model of a School that united knowledge with ethics and community; elements which continue to inspire us even today.

(*) Some ancient sources disagree on the exact date of his birth. If we express 580 in ancient Greek numeration, since $500 = \Phi'$ and $80 = \Pi'$, then 580 becomes $\Phi\Pi'$. Thus, if one accepts the idea of the soul choosing its incarnation, one may poetically say that Pythagoras *chose* to be born in a year composed of Φ (the golden ratio) and Π (the ratio of a circle's circumference to its diameter). Thus the famous phrase «ΑΥΤΟΣ ΕΦΑ» (HE said so") may be extended (making a pun in Greek language) to ...«ΑΥΤΟΣ απ.ΕΦΑ.σισε» ("HE decided").

Pythagoras Today

Moving to the “today,” we can see that Pythagoras’ influence does not belong merely to history. Rather, it permeates modern science, art, and education.

In mathematics, the Pythagorean Theorem continues to be taught as one of the foundations of geometry and finds applications in physics, engineering, computer science, and even music theory. Its three-dimensional form, connected to spatial distances, is used in fields ranging from architecture to cosmology.

Pythagorean Theorem (2-D)

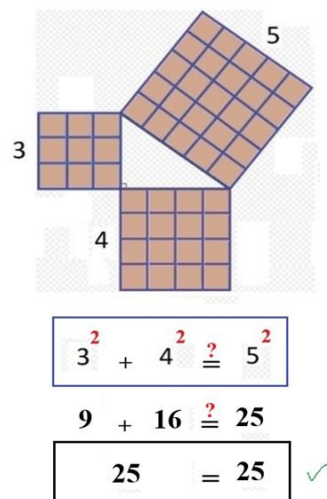


Figure 3: Pythagorean theorem (two - dimensional)

In physics, the idea that nature expresses itself through mathematical relations remains central. From Galileo and Kepler to Einstein, the “language of numbers” remains our path to understanding the universe. The concept of the “harmony of the spheres” finds modern parallels in studies of symmetry, quantum physics, and cosmology.

In music and the arts, the relationship between number and aesthetics continues to inspire creative work — from the Renaissance to contemporary electronic music.

Even more importantly, in education. The emphasis Pythagoras placed on the unity of knowledge and ethics, on the learning community, and on the interplay of theory and practice reminds us today of the need to move beyond fragmented instruction. Modern interdisciplinary and holistic approaches, such as STEAME (Science, Technology, Engineering, Arts, Mathematics, Entrepreneurship) and the EPII-STEAME model [Επιμέλεια (Logistics), Παιδεία (Cultivation), Παιγνιοποίηση (Gamification), Περιβάλλον (Environment) and Ιστορία (History) in STEAME], draw inspiration from this paradigm.



Figure 4: EPII-STEAME

In other words, Pythagoras is not only a historical figure; he is a *contemporary* one, because his ideas continue to guide thought, science, art, and education.

Philosophical Interpretation of the Pythagorean Theorem

The Pythagorean theorem as symbolic harmony expresses the harmonious composition of two forces (the perpendicular sides) into a higher one (the hypotenuse).



Figure 5: Right Angle Triangle

The Pythagoreans did not see mathematics simply as a technical knowledge. They saw the very structure of the cosmic order: How the multiple and diverse can be united in proportion and harmony.

The Platonic Tripartite Soul: Plato mainly in the "Republic" and in the "Phaedrus" speaks of

three parts of the soul: Logistikon = the rational part (mind, reason), Thymoeides = the spirited part (emotions, will, courage, passion) and Epithymetikon = the appetitive part (desires, drives, needs).

Justice and virtue in general arise when these three parts are in harmonious proportion: Rational (mind, reason) guides, Spirited (emotions, will, courage, passion) supports and Appetitive (drives, needs) obeys.

Geometric proportion and mental harmony: If we look at the Pythagorean Theorem symbolically, we can relate it to the Platonic tripartition of the Soul.

The smallest side (a) corresponds to the appetitive principle. The “earthlier” force: body and its needs. The middle side (b) corresponds to the spirited part the intermediate force of the soul: motion and passion. The hypotenuse (c) expresses the rational principle (harmonious synthesis of the two previous ones). It unifies, regulates, and brings measure to the soul.

Area = content. By calculating the area, I recognize and evaluate my feelings and desires. This way I perceive strengths, weaknesses, needs and how to lead myself to their combination (harmony). The hypotenuse (c) expresses the rational (harmonic composition of the previous two). It unifies, regulates, and gives measure to the soul.

Plato said that the two irrational parts of the soul (which are our driving force) correspond to two horses: the one, of white colour, represents thymoide (emotions) and the other one, of black colour, represents desires (wishes, pleasures, urges).



Figure 6: The couple of horses - White (emotions) and black (desires)

Our emotions create various desires. Then, if a desire is fulfilled or not, additional emotions arise, which in turn generate other desires, and so on. In this way there is a continuous circle of interactive emotions and desires, which sometimes has as a result overthinking, negative behaviour and negative actions.

That is why, in such cases, to limit the range of α and β , when the two horses calm down, we study them (we raise the two perpendicular sides in the square).

and then put them together (we add the two areas (9 +16) into another equivalent (empty) space - square (with area 25) where they fit harmoniously. Thus, there is no longer any opposition.

Logic (the charioteer) corresponding to the hypotenuse balances the two illogical parts. Thus, from the everyday/human stage, man is elevated to the philosophical stage: thus occurs the union with his heroic element (with his Higher Self).

In a triangle, the hypotenuse does not exist without the two perpendicular sides (it transcends them through the order of ratio). In the soul, the rational part does not abolish the other two (it composes them into a higher harmony).

The cosmological dimension: Plato (427 - 347 BC) deeply influenced by Pythagoras (580 - 496 BC). He believed that the soul of the world and every human soul are organized according to numerical and geometrical ratios (see *Timaeus* 35a–36b).

Thus, the Pythagorean theorem expresses the same law of harmony that governs the World and the Soul, the way in which multiplicity (two dimensions) ascends to unity (third dimension).

In summary: The Pythagorean theorem in a Platonic philosophical interpretation reveals the unity of the three powers of the soul through the harmonious composition of reason that is the balance between the body, passion, and mind: geometric image of psychic and cosmic justice.

The Timeless Power of the Theorem: that is why the Pythagorean theorem has never faded from the memory of the ages. Because it speaks not only to logic but also to the soul. It speaks of the way a person can unite their opposites, measure their excesses, and create balance within themselves. This is the reason why even today, every child who learns that 'the square of the hypotenuse is equal to the sum of the squares of the two perpendicular sides,' learns, without knowing it, something deeper: that the world is made with law, order, and beauty.

The Pythagorean theorem is not just a matter of mathematics. It is a bridge between number and spirit. It is the geometry of the soul and of the world. It calls us to seek within ourselves the harmony of the triangle - to find the point where the mind guides, passion inspires, and desire serves. Pythagoras: everything is number. Plato: everything is the soul seeking harmony.

When you understand the Pythagorean theorem, you have not merely learned geometry; you have understood the way spirit meets the world.

Pythagorean Theorem (1-D)

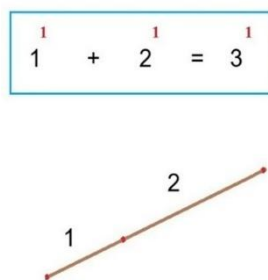


Figure 7: Pythagorean Theorem (one dimensional)

Further to the very well-known Pythagorean theorem, when we consider the simple equation $1 + 2 = 3$ and analyse it philosophically, one of the possible interpretations is the following:

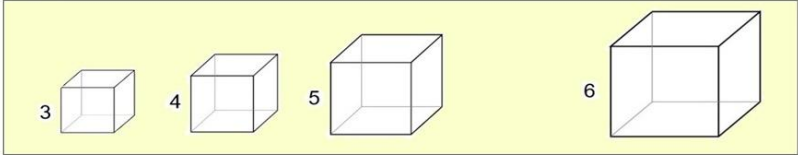
According to Pythagoras, 1 and 2 are indeed numbers and at the same time they are also basic universal principles: (a) One («EN») as a principle and "Monas" ("unit") as the number "1" and (b) the indefinite duo "2" as a principle and also, as a number, the definite duo "2". Number "3" is the first number which has an internal very important force/energy (since it encompasses within itself the two principles, the unit («EN») and the indefinite duo and also the two numbers "1" and "2").

Now, the two universal principles (a) "Monas" («EN») and (b) indefinite duo, are related with the status before Titanomachy, when Saturn – Kronos («EN», acting like a magnetic mono-pole) was the ruler of the Universe, restricting the transformation of the huge concentrated – autonomous Energy and thus restricting progress. The indefinite duo (2) and the definite duo (2), acting like a magnetic di-pole, have to do with the result of Titanomachy, when the reins are in Zeus/ Dias hands (the moment of change of Power maybe correspond to the Big Bang and the beginning of Life).

Three – dimensional "Pythagorean" theorem

The two – dimensional Pythagorean theorem, which has to do with the relationships between tree squares $3^2 + 4^2 = 5^2$ can be extended to the three – dimensional "Pythagorean" theorem, which has to do with the relationships between four cubes as follows: $3^3 + 4^3 + 5^3 = 6^3$

Pythagorean Theorem (3-D)

$\overset{3}{3} + 4^{\overset{3}{}} + 5^{\overset{3}{}} = 6^{\overset{3}{}}$

$3 \times 3 \times 3 + 4 \times 4 \times 4 + 5 \times 5 \times 5 = 6 \times 6 \times 6$
$27 + 64 + 125 = 216$
$216 = 216$

✓

Figure 8: Pythagorean Theorem (three - dimensional)

The sum of the cubes $3^3 + 4^3 + 5^3$, which is equal to 216, when doubled gives 432 and when halved it gives 108. Both these numbers are mystically known, since the ancient times, to have (substantial or coincidental?) significance in relation to arithmetic, geometry, astronomy, universal laws, religion, mythology and more. Some examples are listed below:

(a) Number 432:

Number 432 and Music

432 Hz is a frequency measured in Hertz (Hz), which denotes cycles per second. In musical tuning, this frequency is applied to the note A4, one octave above middle C. While the standard tuning for A4 in modern Western music is 440 Hz, 432 Hz is considered by some to be more in tune with natural vibration patterns, the Earth's resonance, and the golden ratio. But the intrigue goes beyond tuning. The number 432 is a mathematically rich and symbolically potent number, appearing in various ancient traditions, texts, and cosmologies.

Number 432 and time

(a) Number of seconds during a 12 hours day = 43,200

(b) Number of seconds during a 12 hours night = 43,200

Mathematical Properties of 432

432 is divisible by many whole numbers:

1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, 27, 36, 48, 54, 72, 108, 144, 216

This divisibility makes 432 harmonically resonant and easy to integrate into various systems of measurement, rhythm, and proportion.

432 is the product of $2^4 \times 3^3$:

This mathematical factorization shows the combination of two foundational primes used extensively in ancient architecture and numerology.

432 squared equals 186,624:

This number is remarkably close to the speed of light in miles per second (186,282), a correlation often highlighted in metaphysical discussions -even though it may be coincidental.

25,920 the number of years in the Precession of the Equinoxes

Divide this by 60 and you get 432. The precession cycle is significant in ancient astronomy and Vedic cosmology.

These numerical connections suggest a deeper coherence between 432 and the rhythms of nature and cosmic cycles.

(b) Number 108

The Number 108 and astronomical dimensional relationships:

- (a) Distance from Earth to Sun = $108 \times$ diameter of the Sun.
- (b) Distance from Earth to Moon = $108 \times$ diameter of the Moon.
- (c) Diameter of the Sun = $108 \times$ diameter of the Moon.

Number 108 and speed of Light

If we express speed of light in kilometres per hour we get: Speed of light = 1.08 billion km/h.

Mathematical Properties of 108

In geometric terms, it is a natural division of a circle ($108=36+72=9 \times 12$). The number 108 is known to be referring to spiritual completion, and it is no surprise that the early Vedic sages were renowned mathematicians. In their number system, 108 is a Harshad Number, an integer divisible by the sum of its digits. Harshad in Sanskrit means “joy-giver”. The number 108 was the number of choices for this simple reason: In Vedic traditions, 108 represent the whole of existence. The equation for the golden ratio is $2\sin(108^\circ/2) = \Phi$

The Number 108 and Indian tradition:

In Indian tradition there are 108 sacred places

The Number 108 in the Body

According to Chinese and Indian Martial Arts: Marma Adi and Ayurveda, there are 108 pressure points in a human body. There are said to be 108 energy lines, or nadis, converging to form the heart chakra. Marma points are like Chakras or intersections of energy with fewer converging energy lines.

The Number 108 and the Sri Yantra

On the Sri Yantra, the Marmas have 54 intersecting energy lines where three lines intersect. Each has feminine, or Shakti, and masculine, or Shiva, qualities. $54 \times 2 = 108$. Therefore, there are 108 points that define the human body and the Sri Yantra or the Yantra of Creation. The same rule is observed in the Sanskrit language, with its 54 letters, both representing the two genders and they are also called Shiva and Shakti respectively; again, $54 \times 2 = 108$.

The Number 108 and Homer

(a) Iliad: the total number of Greek ships that sailed to Troy were 108×11 (1188).

Note: The relevant ancient sources refer to 1186 ships. The other two "missing" ships, according to the myth, were actually sent by Cyprus' king Kyniras, who promised Agamemnon to send 50 ships but finally fooled him: he sent only 2 ships and also sent another 48 miniature ships made of clay! That is why the Cyprus contribution was not registered in the official list.

(b) Odyssey: Odysseus, when he arrived at his Palace in Ithaca, killed all the suitors (108). Principles (a) «EN» and (b) indefinite duo (2) when they are now combined give as a result the number 3. The internal and philosophical Energy of triad (3), which we also have inside us and we must find the way to develop, if we want to give to our life meaning and the target which it needs. So as to utilise the reason of our existence, in the best possible way based on ancient Greek philosophy and the existing universal state.

Number 108 and the recently unveiled bust of Pythagoras at Cyprus University of Technology

The cement base of the bust (*) of Pythagoras (which was unveiled *at the Cyprus University of Technology* during the works of the Conference "The Life and Contribution of Pythagoras to Mathematics, Sciences and Philosophy", October 3-4, 2025 <https://pythagoras-conference2025.cut.ac.cy>) has dimensions 32 x 25 x 135 centimetres. This was designed by the sculptor of the bust Panayiotis Pasantas (of the Faculty of Fine and Applied Arts of Cyprus University of Technology) after considering various other possible dimensions, related to various factors. By coincidence (or ...instructions from Pythagoras spirit?) the volume of the cement base results to be 108,000 cubic centimetres.

(*) The bust of Pythagoras was designed to intentionally "hide" some "secret" dimensional relationships resulting in golden (Φ) and π ratios. Also, on the two sides of the bust are demonstrations of the three Pythagorean theorems (1-D, 2-D and 3-D) as well the Pythagorean helix on the back.



Figure 9: Pythagoras' bust in Limassol at the Cyprus University of Technology
(Front yard of Andreas Themistocleous building in Athinon street)

The Number 108 in the Stars

The earth cycle is supposed to be of 2160 years = 20×108 . The distance between the Earth and Sun is 108 times the diameter of the Sun. The diameter of the Sun is 108 times the diameter of the Earth. The distance between the Earth and Moon is 108 times the diameter of the Moon. There are 9 planets in the Solar System and 12 segments of the Zodiac. $9 \times 12 = 108$.

There are 12 constellations and 9 arc segments: 9 times 12 equals 108. The 9 planets travelling through the 12 signs constitute the whole of existence. $9 \times 12 = 108$. The 27 lunar constellations spread over the 4 elements – fire, earth, air, water or the 4 directions – north, south, east, and west. This also constitutes what is referred to as the whole of existence: $27 \times 4 = 108$.

The Number 108 in Spiritual Practice

In Kriya Yoga, the maximum number of repetitions allowed to be practiced in one sitting is 108. Also, 108 Sun Salutations in yoga practice are often used to honour change. For example, the change of seasons, or at a time of tragedy to bring peace, respect and understanding. It is said that if one can be so calm in meditation practicing pranayama to have only 108 breaths a day that enlightenment will come. There are said to be 108 types of meditation. Some say there are 108 paths to God. Indian traditions have 108 dance forms.

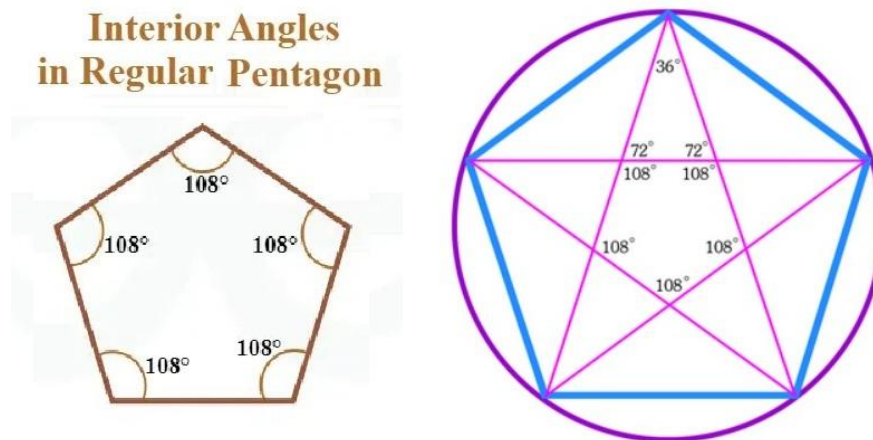


Figure 10: Relation of number 108 with Pentagon and Pythagorean Pentalfa

Human body, mind and spirit in the Eastern tradition

In the Eastern tradition and in yoga, the human body is considered to have seven main energy centres known as chakras. Each energy centre corresponds to a specific area of the body and is associated with important psychological, emotional, and spiritual functions.

The harmonious flow of energy through these centres contributes to a person's balance, while blockages can manifest as anxiety tension overload or alienation from oneself. Caring for and balancing the chakras is achieved through practices such as breathing meditation yoga sound/singing (mantra) and mindful way of living.

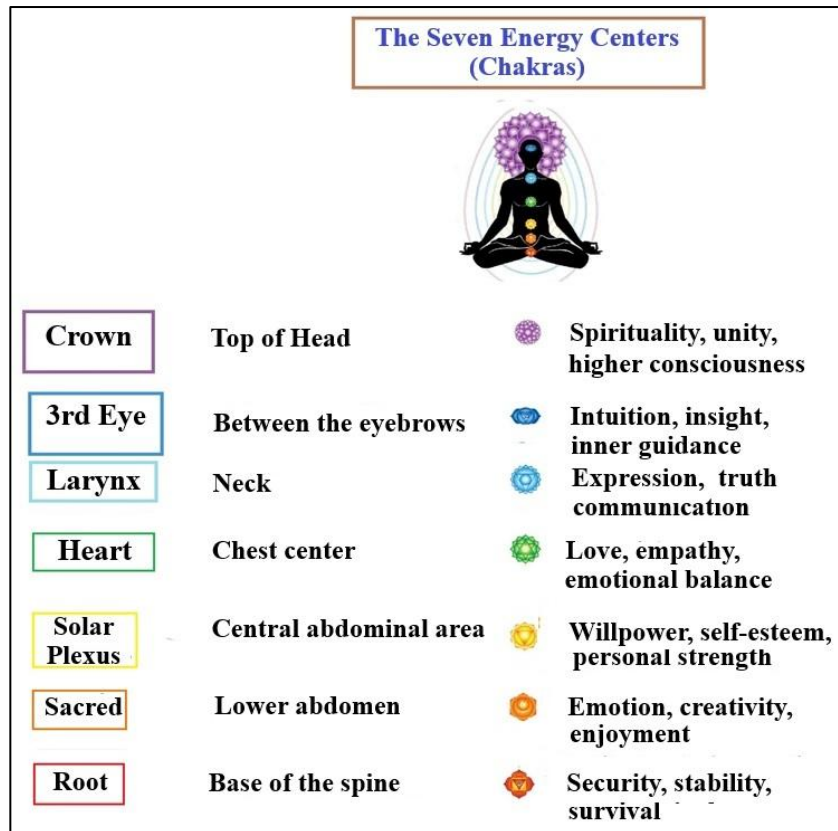


Figure 11: The seven Energy Centres (Chakras)

When someone is on the "frequency" of survival (of stress), they feel fear, pain, anger, hatred, insecurity, etc., which are not the opposites of Love. It is simply the absence of it. That is why it is important for someone to prioritize managing their energies and the focus of their attention, the mindful way of life.

The distancing from love due to some event or trauma (childhood or otherwise) makes us see love through a 'different' lens. If we have any feeling for ourselves, positive or negative, we feel the same in our relations with others. Whoever is angry with themselves is also angry with others. Whoever loves themselves also loves others.

Pythagoras Tomorrow

Our time is marked by rapid technological progress, ethical dilemmas surrounding artificial intelligence, social inequalities, and the need to build a new relationship with nature and with humanity itself.

In this context, Pythagorean teachings may guide us in three ways:

1. Harmony and unity of knowledge:

In a world where sciences often operate in isolation, the Pythagorean example calls us to synthesise - to see physics alongside philosophy, mathematics alongside aesthetics, technology alongside ethics.

2. The pedagogical dimension:

Just as the *Omakoeion* was not only a place of learning but a way of life, so the School of the Future can become a space where education integrates values, cooperation, and responsibility.

3. Ethics and number:

If number is a measure of harmony, then we must seek balance between technological progress and humanity, between innovation and responsibility.

Over the past 30 years I am collaborating in Cyprus with The Ministry of Education, the Pedagogical Institute, Universities, schools, teachers and facilitators on school-based initiatives designed to foster empathy, expression, and co-creation among students aged 5 - 11 and 12 - 17.



Figure 12: Happiness in Teaching and learning

I have the chance and the pleasure to present, in schools of all levels, both in Cyprus and abroad, various topics of the curriculum (using Alternative Teaching and Learning Methods, through cartoons, games, fairytales, music etc.) like Mathematics, History, Electronics, Mythology, Telecommunications, Physics, Robotics, Greek Language, Artificial Intelligence, introducing children and adults to Cartoons, Leadership and Educational Administration, Positive Verbal Communication, Alternative Teaching and Learning Methods, and more.



Figure 13: "Cartoons as teaching tools"

A list of some of the relevant projects or activities follows:

- (1) **I Love my self**
- (2) **Introduction to Cartooning / Caricature**
- (3) **School Wall - Caricatures**
(Collaborative mural-style cartoon creations in school spaces)
- (4) **Cypriot–Homeric Words and Expressions Illustrated with Sketches and Cartoons**
- (5) **Games of Knowledge and Joy**
An indirect approach to reducing School Bullying
- (6) **EIII–STEAME**
(Creative integration of STEAME subjects: Science, Technology, Engineering, Art, Mathematics, Entrepreneurship)
- (7) **Music and Mathematics**
- (8) **LEXIC.OM.ORPH. ON**
(Creative Lexicon of Shared/Related Words:
A playful exploration of Greek etymology and morphology)
- (9) **Human and Artificial Intelligence**
- (10) **How to Comfortably Succeed in Any Examination**
- (11) **Intercultural Education**
- (12) **“Life-Long Learning, Not Learning by Force”**
In collaboration with parents and the school
- (13) **DE-CLAS-SIFICATION**
(Meaning: breaking rigid age/class barriers; mixing learning groups creatively)
- (14) **Mathematics for Parents**
- (15) **DE-TIK-TOK-SIFICATION**
(Guidance in healthy use of social media, especially TikTok; media literacy and digital well-being)

(16) More Positive Verbal Communication in Everyday Life

(17) Students as Caregivers / Mentors of Younger Children

(18) Electronic Living Library

(Living / interactive library – learning through sharing experience rather than only reading)

(19) Observatory of the Greek Language

conducted by children, educators, and parents of the All-Day School

(20) A School Without a Bell

(Flexible, humane time-flow learning environment).

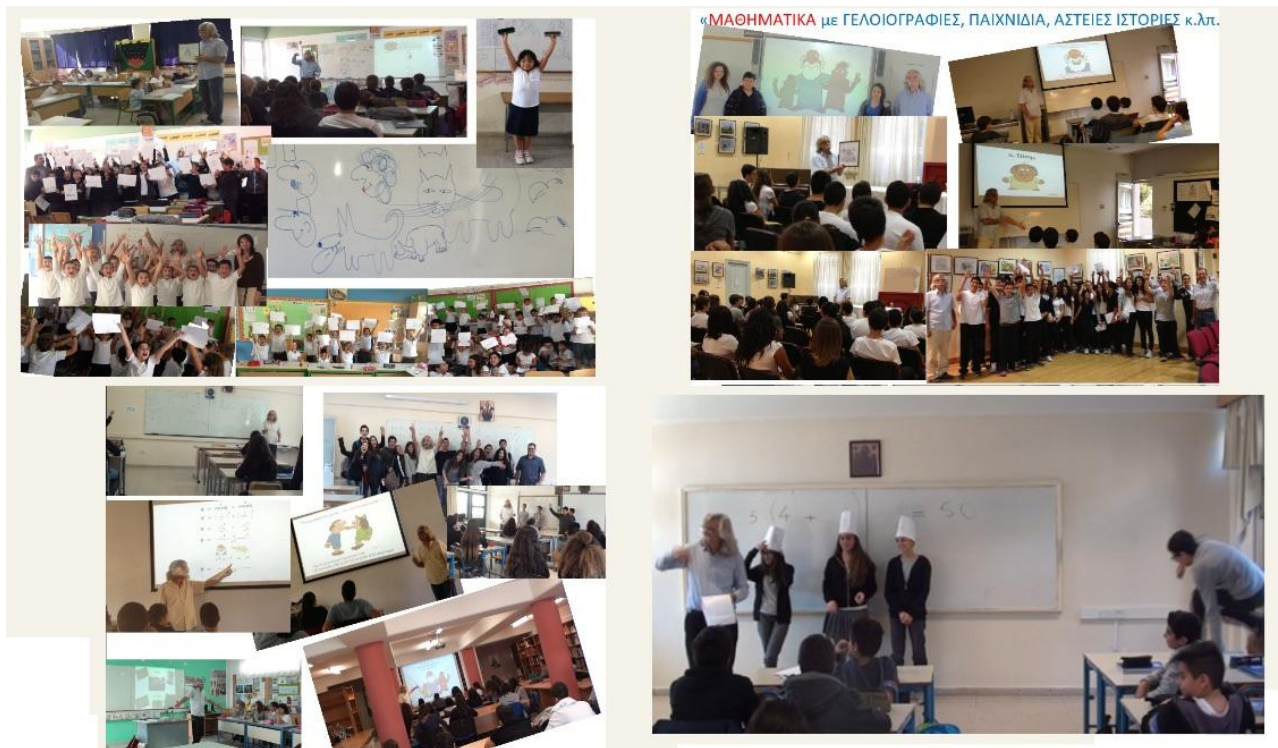


Figure 14: Learning Arithmetic, algebra, geometry, trigonometry using cartoons and gamification



Figure 15: How to Pass any examination easily (rehearsal of the graduation ceremony)

STEAME and Pythagoras

Today the majority of school students worldwide lack competences and skills at the completion of the cycle of their school studies. As one of the key factors that can develop competences and skills in school students, especially the younger ones, is the well-known multidiscipline and mutli-science project-based learning activity STEAME (Science, Technology, Engineering, Arts, Mathematics and Entrepreneurship).

The creation of a STEAM school (with A -arts incorporating culture and communication) is an important step towards the necessary educational redesign, so that through project-based learning activity of school, students will become the magic force for shaping the creators and innovators of the future. The target of the STEAME Project is to develop a model through a STEAME Teacher Facilitators Academies networked through a European Federation of such Academies at local level, which will support change for schools in the future with main actors being the the teacher as the future professionals and experts of learning change and facilitators of competence and skills development in youth.

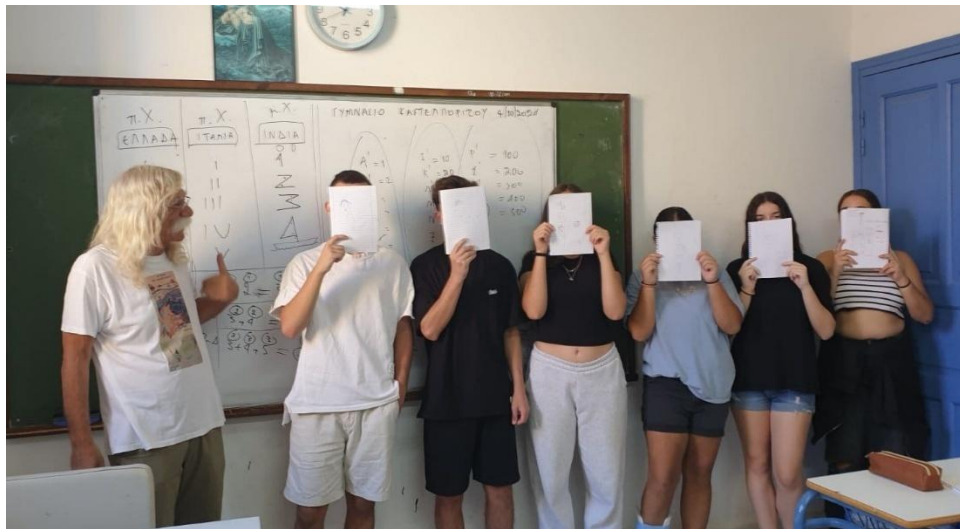


Figure 16: STEAME workshop in the tiny island of Kastellorizo in Greece

Considering Pythagoras' School in Kroton (Omakoeion) it can be said that Pythagoras was the first teacher to apply STEAME approach. His students were indeed learning Sciences, Technology, Engineering, Arts, Mathematics and Entrepreneurship, as well as Astrology, Biology, Nutrition, philosophy and much more.

ΕΠΙΛ-STEAME

Creative integration of STEAME subjects: Science, Technology, Engineering, Arts, Mathematics, Entrepreneurship.

A Good Practice Case: The Circle of Trust and Imagination (Basis is Trust and Imagination)

The MOSSAiC (Matheses-Learning on Saturdays and Sundays: Arts, Imagination, Creativity)

Example:



Figure 17: STEAME anywhere A

Five foundational pillars or building schools that empower both students and teachers to thrive:

Five Pillars for an Empathic School Culture

1. Belonging Before Performance – Emotions first, outcomes second. Students learn best when they feel they belong.
2. Project-Based Learning – Link education to real-world challenges like sustainability and civic engagement.
3. Arts and Role-Play – Drama, storytelling, and visual expression open students to multiple perspectives and empathy.
4. Mindfulness and Metacognition – Teach students to notice and regulate their thoughts and emotions.
5. Shared Leadership – Involve students in shaping school culture, policy, and curriculum.



Figure 18: STEAME anywhere B

Methodology:

1. Opening Circle: Emotional check-in to build trust and emotional awareness

2. Imaginative Prompt: Offering a creative and open-ended question to ignite imaginative thinking

3. Co-Creation Task

Students engage in a hands-on activity: there is no right answer but only authentic expression and shared discovery

4. Closing Reflection

Students share something they noticed, learned or appreciated (peer acknowledgments are encouraged)



Figure 19: STEAME anywhere C



Figure 20: STEAME anywhere D

The School as a Safe Space

“Safe school” is a space where students feel they can speak and be heard. Mistakes are treated as learning opportunities diversity is respected and bullying (verbal, digital, or social) is confronted with clarity and compassion.



Figure 21: STEAME anywhere E

The School as a Creative Space:

Students are encouraged to imagine, take risks, explore uncertainty, co-create meaning with others. Schools must be rooted in **empathy**: The capacity to understand and feel with others Empathy is a survival skill for the 21st century for democracy, peace, innovation, sustainable living.



Figure 22: Creativity anywhere A

The three-dimensional version of the Pythagorean Theorem, revealing insights related to the speed of light, planetary distances, and musical scales, can function as a symbol of how distances between sciences, peoples, and cultures may be bridged.

Pythagoras “yesterday, today, and tomorrow” is not just an object of study - it is an *invitation*. An invitation to think more holistically, to unite number and harmony, science and art, knowledge and education, past and future.

Pythagoras reminds us that number is not a cold calculation. It is a bridge to the harmony of the world.

If we draw inspiration from his thought, we may create a future where care, education, ethics, playfulness, history, science, technology, engineering, the arts, entrepreneurship, and knowledge co-exist in balance.

Number, Harmony, Humanity, Love - the path.

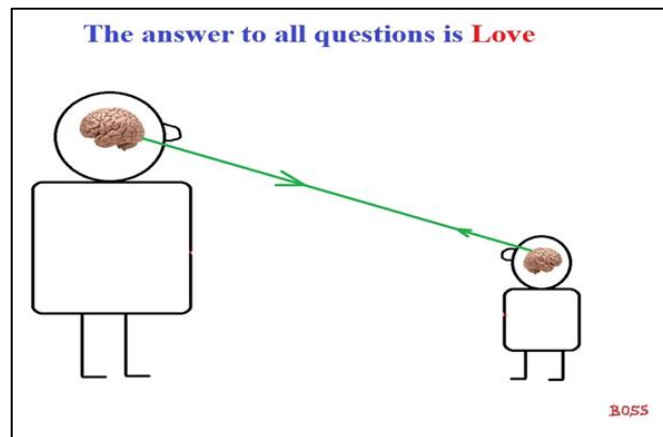


Figure 23: The answer to any question is...

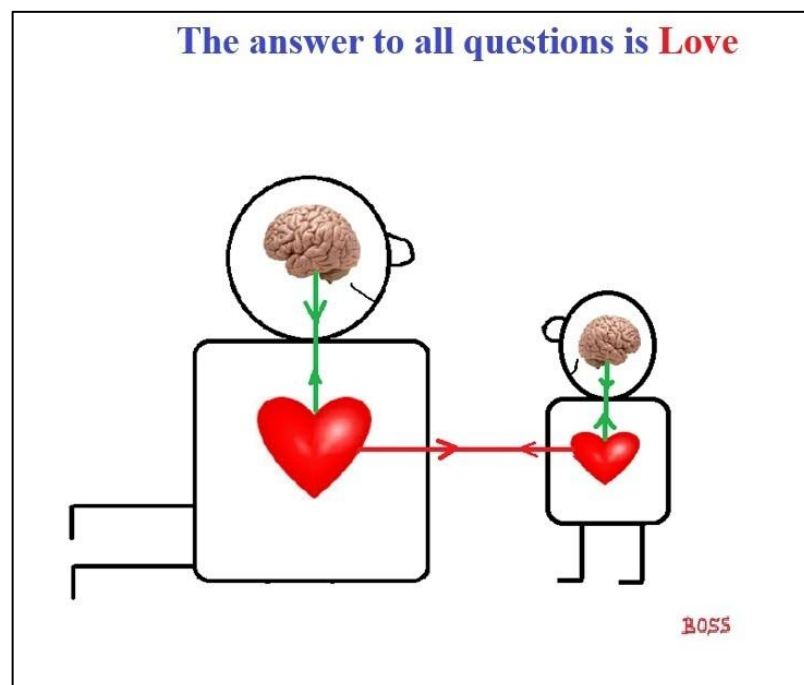


Figure 24: The answer to any question is Love

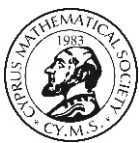
COMPARISON: Plato's Tripartite Soul and Seven Energy Centers (Chakras)			
PLATONIC STRUCTURE	FUNCTION	Corresponding CHAKRAS	COMMON GOAL
Rational	Cognition JUDGMENT Guidance	Crown Third Eye Throat	Wise guidance of the other two
Thymus	COURAGE Emotional Strength	Heart Solar Plexus	Cultivation of healthy SELF- RESPECT
Appetitive	SURVIVAL Desire Pleasure	Sacred Root	Harmonization of basic needs

TABLE 1: Comparison of Plato's Tripartite Soul and the Seven Energy Centers (Chakras)

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The "Semicircle", the first Pythagorean school in Samos. The school's subjects and who were its potential students

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SUMMARY

Historical accounts indicate that Pythagoras was born on Samos around 580 BC and passed away circa 497 BC in Metapontum, Magna Graecia, at the age of 83. His biographer, Iamblichus (V.P. 11), refers to him as the "**long-haired Samian**"—using the term *komētēs* (derived from the Greek word *komē*, meaning "long hair")—noting that his exceptional beauty and intellect garnered admiration from early childhood. He was entrusted to the most eminent teachers of the era, including the philosopher Pherecydes of Syros, as well as Thales and Anaximander of Miletus. After extensive travels through Phoenicia, Egypt, Chaldea, Persia, and India, Pythagoras returned to Samos, which was then under the rule of the tyrant Polycrates. There, he began teaching in an open-air setting known as "**The Semicircle**." The School of Samos is regarded as the first institution of higher education. This academy operated until 530 BC, when Pythagoras relocated to Croton in Southern Italy to establish a new school. According to Justin Martyr, Pythagoras led his school for a total of 39 years: 19 in Samos and 20 in Croton. This article explores the subjects taught at the school, his potential students, and the reasons why, according to tradition, Pythagoras was compelled to teach for a significant period within a cave, known to this day as "**Pythagoras' Cave**."

Introduction

Pythagoras was born around 580 BC on Samos, an island under the patronage of the goddess Hera. In her honor, the Samians erected the **Heraion**—a magnificent temple whose massive marble columns dominated the landscape, visible even from the distant coast of Ionia. The historian Herodotus provides the only mention of Pythagoras's father (*Histories*, IV, 95), identifying him as Mnesarchus. His mother, Parthenis, earned a reputation as the most beautiful woman of Samos. According to Diogenes Laertius (citing Aristippus of Cyrene in *Lives*, VIII, 21), both parents descended from the hero Ancaeus, the son of Zeus and founder of Samos. A compelling tradition surrounds his naming: when Mnesarchus and his wife visited the Oracle of Delphi, the Pythia foretold her pregnancy. Following this divine announcement, the parents named the child Pythagoras—meaning "the Pythian messenger"—and his

mother adopted the name Pythais. As the son of a wealthy merchant, gem-engraver, and potter, Pythagoras grew up in affluence. This privileged background allowed the young Pythagoras —already renowned for his striking appearance and intellect— to study under the most distinguished teachers and philosophers of his time.

Pythagoras: Teachers and Studies

Among Pythagoras's earliest mentors was **Pherecydes of Syros**, who was influenced by Orphism, Thales's cosmological theories, Hesiod's theogony, and the wisdom of ancient Egyptian priests. Pherecydes taught his disciples in a cave; a site that still bears his name in the north-eastern part of Syros. Historians credit Pherecydes with measuring the Sun and consider him the inventor of the first **sundial**. According to Cicero (*Tusc.*, I 16), he maintained that human souls are immortal. In the 6th century BC, when Polycrates, the tyrant of Samos, conquered Syros, Pherecydes moved to Samos and imparted his influential ideas to Pythagoras. Pythagoras was profoundly influenced by these teachings; taking his teacher's example, he would later teach his own students in a cave on Samos. According to the Greek philosopher and mathematician **Dicaearchus** (4th century BC) in the work *Porphyry*, P B55, 56, Pythagoras attended Pherecydes during the final moments of his life. Furthermore, **Aristoxenus of Tarentum** (370–304 BC) states in *Diogenes, Lives I, 118*, that Pythagoras buried his teacher on the island of Delos.

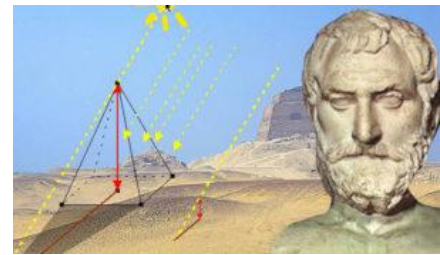
Due to the close proximity of Samos to the eastern coastal city of Miletus, we can assume —without being certain— that the famous **Thales of Miletus** (640–546 BC) was also a teacher of Pythagoras. Thales, one of the Seven Sages of antiquity, was a mathematician, physicist, astronomer, engineer, and meteorologist, as well as the founder of the Ionian School of physics and philosophy in Miletus. Thales was among the first Greek thinkers to influence the intellectual timeline for the next 1,000 years. The young Pythagoras studied under the "Great Master," who instilled in him a passion for philosophy and mathematics. With vast and pioneering knowledge for his time in physical science, geometry, and mathematics, Thales served as a beacon and a role model for all scientists of subsequent generations. As his student, Pythagoras can be considered the successor to the geometric ideas of the Milesian philosopher. From Thales, Pythagoras would learn:

- **Prudence in the allocation of time:** that is, to act only after careful study and with wisdom regarding the time at one's disposal.
- **Temperance (Self-restraint).**
- **True Science**

One of the most significant mathematicians of the 20th century, **Bertrand Russell**, stated that "Western philosophy begins with Thales," as Thales was among the first scientists to engage with **Logic**, which was Russell's primary field of study.

In Geometry, Thales is considered the first to conceive the proof of geometric propositions using the method of "**reductio ad absurdum**" (proof by contradiction) and introduced the use of hypotheses in the search for truth. Thales gave new meaning to the geometric figure, establishing it as an object of study and mathematical thought; previously, the figure served only as a means to record numerical values such as length, quantity, etc. According to **Proclus**, the theorems and problems attributed to Thales and taught to Pythagoras are:

- **The circle is bisected by its diameter:** *"Το μεν ουν διχοτομίσθαι τον κύκλον, υπό της διαμέτρου, πρώτον Θαλήν, εκείνον αποδείξει, φασίν."*
- **An angle inscribed in a semicircle is a right angle** (knowing that the sum of the angles of a triangle is equal to two right angles).
- **The base angles of an isosceles triangle are equal:** *"Τωι μεν ουν Θαλήι τωι παλαιώι πολλών τε άλλων ευρέσεως ένεκα και τούδε του θεωρήματος χάρις. Λέγεται γαρ δη, πρώτος εκείνος επιστήσαι και ειπείν, ως αρα παντός ισοσκελούς αι προς τη βάσει γωνίαι ίσαι εισίν."*
- **When two lines intersect, the vertical angles are equal:** *"Τούτο τοίνυν το θεώρημα δείκνυσιν, ότι δυο ευθειών αλλήλας τεμνουσών, αι κατά κορυφήν γωνίαι, ίσαι εισίν, ευρημένον μεν, ως φησίν Εύδημος, υπο Θαλού πρώτου."*
- **Two triangles are congruent if they have one side and the two adjacent angles equal** (In this way, he calculated the distance of a ship from the harbor, a technique later used by the Romans).
- **Equiangular (similar) triangles have proportional sides:** thus, using the length and the shadow of a staff, he calculated the height of the Great Pyramid of Egypt.
- **The famous theorem that bears his name:** if two random lines are intersected by a set of parallel lines, the segments between the parallels are proportional.



In Miletus, Pythagoras also met **Anaximander of Miletus** (611–547 BC). From Anaximander, he learned the theory of "**evolution**" (**anelixis**). The core idea of this theory is that the world and all phenomena originate from a primary matter called the "**Apeiron**" (the Boundless), which is in constant transformation and motion. The main elements of this theory are:

- Matter is infinite and limitless.
- All things originate from this matter through successive changes and mixtures.
- Motion and change are fundamental elements of nature.
- The succession and continuous evolution of elements lead to the creation of the world.

Anaximander believed that:

- The Earth is suspended in space and rests on nothing; it remains buoyant because it is equidistant from everything.
- The shape of the Earth is cylindrical, similar to the drum of a column.

- The first animals were born in the sea and, over time, adapted to the land.

In Miletus, Pythagoras was also taught by **Anaximenes** (585–528 BC), the third Milesian sage and student of Anaximander. The theory of Anaximenes involves the successive transformation of matter through **condensation and rarefaction**. He also believed that:

- ✓ The Earth is created through the condensation of air and is supported by it.
- ✓ The Sun and other celestial bodies consist of vapors from the earth; they are flat like leaves and rotate around the Earth.
- ✓ The shape of the universe is hemispherical.
- ✓ Rain, clouds, and other meteorological phenomena are explained through the condensation and rarefaction of air.

After completing his studies in Ionia and Samos, and with the support and recommendations of the tyrant Polycrates, Pythagoras continued his education outside of Greece in the then-known world. The philosopher **Iamblichus** (250–325 AD), biographer of Pythagoras, mentions in his book *On the Pythagorean Life* that Pythagoras traveled to Samothrace, where he was initiated into the **Cabeirian Mysteries** at the Sanctuary of the Great Gods. The Cabeiri were deities of an ancient mystery cult that first appeared in Lemnos, Imbros, Samothrace, and Thebes, later expanding to Macedonia-Thrace and the Aegean. In the Cabeirian Mysteries, initiation focused primarily on genesis (creation) and less on death, somewhat analogous to Christian baptism. The initiation consisted of five parts: **purification** (*katharmos*), **delivery of the rite**, **supervision** (*epopteia*), **binding of wreaths**, and **perfection and bliss**.

Following a recommendation from the tyrant Polycrates of Samos to the Egyptian Pharaoh Amasis II, Pythagoras went to **Egypt**, where he remained for 20 years studying Egyptian philosophy and mathematics, which at the time were considered sacred sciences. Numbers in ancient Egypt held primarily practical and symbolic value, without deep study into number theory or the properties derived from them. Beyond practical uses for construction (pyramids, temples) and trade, numbers were linked to divine order, cosmology, and the functioning of the universe. Pythagoras also studied astronomy and geometry in Egypt.

The Egyptians used and developed geometry for the construction of pyramids and other monuments, building construction, land distribution, and the management of agricultural resources following the flooding of the Nile. Sources such as the "**Rhind Papyrus**" and the "**Moscow Papyrus**" demonstrate the Egyptians' understanding of the concepts of area and volume for geometric shapes, including cylinders and pyramids.

In the geometric section of the Rhind Papyrus, there is no trace referring to knowledge of the Pythagorean Theorem. Its geometric problems involve the numerical calculation of surface areas or volumes of solids. The area of a circle is calculated by considering its value.

In the Moscow Papyrus, there is a problem involving the calculation of the surface area of a hemispherical vessel. Thus, the Egyptians were able to measure the surface of a sphere. Indicative examples of geometric problems include the following:

- **Problem 50 of the Rhind Papyrus:**

"Example of a round field with a diameter of 9. What is its area?"

The calculation is described as follows:

Subtract 1/9 of the diameter. Therefore, the remainder is 8.

Multiply 8 by 8, which equals 64.

Thus, the area of the field is 64.

In other words, the scribe of the papyrus uses the formula:

$$A = \left(d - \frac{d}{9}\right)^2 = \left(\frac{8}{9}d\right)^2 \quad \text{όπου } d = \text{diameter}$$

The formula we use today to calculate the area is:

$$A = \pi R^2 = \pi \left(\frac{d}{2}\right)^2 = \frac{\pi}{4}d^2$$

Comparing the two formulas, we find that:

$$\frac{\pi}{4} = \frac{64}{81} \quad \text{or} \quad \pi = \frac{256}{81} = 3,16049 \dots$$

- **Problem 14 of the Moscow Papyrus:**

In this problem, the volume of a truncated pyramid (frustum) is calculated.

"The problem provides the data that a pyramid has been truncated so that the top surface is a square with side length 2, the bottom surface is a square with side length 4, and the height is 6. What is the volume of the pyramid?"

The calculation is described as follows:

Calculate with the 4 by squaring it. Result: 16.

Double the 4 (multiply 4 by 2). Result: 8.

Now calculate with the 2 by squaring it. Result: 4.

Add the 16, the 8, and the 4. Result: 28.

Calculate one-third of 6.

Result: 2.

Multiply the 28 by 2. Result: 56.

Indeed, if we let α be the side of the top base, β the side of the bottom base, and h its height, then following the previous algorithm, we arrive at the correct result:

$$V = \frac{h}{3}(a^2 + a\beta + \beta^2)$$

After Egypt, Pythagoras went to Babylon and for 12 years was taught by the wise men of Persia. He particularly studied the theory of numbers and the functioning of the Universe but also learned the

application to Geometric problems of the practical use of his "Pythagorean theorem". The Babylonians had also developed Algebra and could solve a system of linear equations, with two or more unknowns, in a way that we in modern times follow. The Austrian-American mathematician and historian of science Otto Eduard Neugebauer (1899-1990) studying clay tablets discovered that the Babylonians knew much more about Mathematics and Astronomy than had previously been realized. The translations of all the texts have been included in the monumental work of *MKT (Mathematische Keilschrift-Texte) Volumes I-III*. We mention two examples:

• **Example 1 (MKT, Vol. III, page 6)**

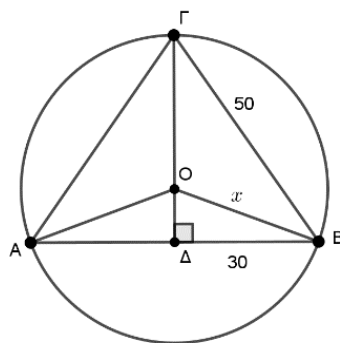
If the side (the length of the side) of a square is subtracted from the square (the area of the square) the result is 14,30. What is the length of the side?

This asks for the value of the variable x for which $x^2 - x = 14,30$ applies

Describing the Babylonian Solution in Modern language we will have (in the Babylonian sexagesimal system the numbers are):

- ✚ Let $x - 1 = y$. When $x - y = 1$ and $xy = 14,30$ (because $x^2 - x = x(x - 1)$)
- ✚ Let $x = a + 0;30$. When $y = a - 0;30$
- ✚ $(a + 0;30)(a - 0;30) = 14,30$
- ✚ $a^2 = 14,30 + 0;15 = 14,30;15$
- ✚ $a = \sqrt{14,30;15} = 29;30$
- ✚ $x = 29;30 + 0;30$
- ✚ $x = 30$

- **Example 2 (Babylonian Geometry)** *The following problem was found on a clay tablet: The following figure is given:*



in which an isosceles triangle $\triangle AB\Gamma$ with $AB = 60$ and $A\Gamma = B\Gamma = 50$. It is required to calculate the radius of the circumcircle of the triangle.

Babylonian Solution:

✚ The Babylonians were able to calculate the side $\Gamma\Delta$ of the triangle $\triangle\Gamma B\Delta$ applying the Pythagorean Theorem. Therefore, they could find that $\Gamma\Delta = 40$.

✚ Let $OB = x$.

✚ By applying the Pythagorean Theorem in the triangle $\triangle MB\Delta$ we find:

$$\begin{aligned}x^2 &= (40 - x)^2 + 30^2 \\80x &= 2500 \\x &= 31\frac{1}{4}\end{aligned}$$

$$O\Delta = 40 - 31\frac{1}{4} = 8\frac{3}{4}$$

After Babylon, some scholars argue that Pythagoras most likely traveled to **India**. He was influenced by the teachings of the **Brahman** wise priests (members of the highest Hindu caste), especially regarding the transmigration of souls (reincarnation). He was also taught the importance of silence, prayer, and other disciplinary exercises and lessons. As a contemporary of **Buddha** and **Confucius**, there are indications that he became acquainted with their philosophies.

After his studies and travels, Pythagoras returned to Samos at the age of 56. He was immediately invited by the tyrant **Polycrates** and is said to have resided in his palace as a guest.

The School of Pythagoras in Samos

Upon his return, Pythagoras made it his life's goal to benefit the youth of Samos. With the consent of Polycrates, he began teaching at a school established in "**Astypalaia**" (modern-day Pythagoreio). He named his school the "**Pythagorean Semicircle**", which had 28 students. This "Semicircle" later served as a model for the construction of the Theater of Dionysus in Athens. Outside Pythagoreio, on a slope overlooking the sea, stood the **Cave of the Nymphs** (possibly the current cave of Panagia Spiliani, See photo). In this cave, it is said that his father, Mnesarchus, a wealthy merchant, had established a sanctuary of Apollo with the inscription: "**To Pythian Apollo.**" («*Τῷ Πυθίῳ Απόλλωνι*»).



the small chapel in honor of the Virgin Mary

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This was Pythagoras' favorite place, where he would spend many hours, day and night, discussing scientific topics with a few elite students. Pythagoras taught at a very high level, applying the **"Egyptian method of teaching,"** which relied on memorization, copying texts, and practical application. His teaching was often beyond the students' capabilities, and he frequently found himself almost alone in the classroom. It is said that he initially paid his students to keep them in the school. Eventually, beyond these difficulties, he faced external friction with the authorities of Samos.

At that time, his former teacher **Anaximander** was living in Samos in his old age. He came into conflict with Pythagoras due to their differing beliefs and teachings, which contradicted Anaximander's own philosophy. Anaximander accused Pythagoras of impiety and incited the people of Samos against him. He managed to convince philosophers and priests that Pythagoras was acting against the Gods of Greece because he taught the existence of a single, unified Deity. Once Polycrates was also convinced of this "impiety," he expelled Pythagoras from the palace and arranged for him to appear before a court on charges of disrespecting the Gods.

Subjects Taught at the Pythagorean School in Samos

The School of Pythagoras in Samos was the first institution of higher learning—one might say the first **University**—where basic sciences and the accumulated knowledge of the Ancient World were taught. The school accepted both male and female students. The curriculum was almost equivalent to modern academic programs, though the teaching method differed.

For first-level students, the **"Akousmatikoi"** (Listeners), the method was "Acromatic" (listening-based). They listened to his doctrines without objection because, as they said, «*Αυτός ἔφα*» (**"He himself said it"**). The **"Mathematikoi"** (Mathematicians) or "Pythagoreans" were the students engaged in true knowledge under the exclusive guidance of Pythagoras.

The Subjects of the Pythagoreans

The core subjects taught were **Philosophy** and **Science**. Philosophy dealt with "knowledge" and the nature of the world of immaterial and eternal things, while Science dealt with the knowledge of material and transient things.

Philosophy:

Although it is difficult to isolate the original core of Pythagoras' teaching, it appears that, besides moral, religious, and political precepts, it included a philosophical worldview with early scientific concepts. The teaching of Philosophy also included the theory of knowledge, the soul, and justice.

Number Theory (Arithmetic)

The number theory taught was the theory of **Natural Numbers**. The oldest part of Number Theory

that we can reconstruct is:

- **The theory of even and odd numbers**

In the ninth book of Euclid's *Elements*, we encounter a succession of propositions (21–34) which have no connection to anything that precedes them. According to **O. Becker**, these propositions—together with proposition IX.36—constitute a truly archaic and formal Pythagorean sample of Mathematics.

The pinnacle of the theory of even and odd numbers is indeed **Proposition 36**, which asserts that numbers of the form:

$$2^n(1 + 2 + 2^2 + \cdots + 2^n)$$

are **perfect**, provided that the factor

$$p = 1 + 2 + 2^2 + \cdots + 2^n$$

is a prime number.

- **Arithmetic Proportions**

The systematic organization of the theory of ratios between numbers and divisibility, contained in the seventh book of Euclid's *Elements*, was developed after the theory of even and odd numbers and is attributed to the Pythagoreans.

- **Solving Systems of First-Degree Equations**

The solution of a specific type of system

$$\begin{aligned} x + x_1 + x_2 + \cdots + x_{n-1} &= s \\ x + x_1 &= a_1 \\ &\dots \\ x + x_{n-1} &= a_{n-1} \end{aligned}$$

is known by the name "**Thymaridas' Bloom**" (*Thymaridas' Epanthema*).

The solution is indeed:

$$x = \frac{(a_1 + \cdots + a_{n-1}) - s}{n - 2}$$

The Neopythagorean Iamblichus, who attributes this solution to the ancient Pythagorean Thymaridas of Paros, further demonstrates that systems of the form

$$x + y = a(z + u), \quad x + z = \beta(u + y), \quad x + u = \gamma(y + z)$$

can be reduced to the previous form. We thus see that the Pythagoreans, like the Babylonians, engaged in solving systems of equations with more than one unknown.

Geometry

In the field of Geometry, the Pythagoreans did not focus on formal proofs or axiomatic foundations in the way Euclid later did. Instead, their primary interest lay in exploring the fundamental properties of plane figures. There are indications that a geometry manual existed under the title "**Inquiry According to Pythagoras**" (pros Pythagorou istoria), which likely served as a collection of written lectures. The Geometry of the Pythagoreans was logically structured; they were familiar not only with the proposition regarding the sum of the angles in a triangle but also with the properties of **alternate interior angles** formed by parallel lines. It is highly probable that they possessed a coherent theory of regular polygons. They certainly knew of the **star pentagon** (pentagram) and the **dodecahedron**. Furthermore, they had discovered that there are only three regular polygons whose angles can perfectly tile the plane around a single point **O** —namely the **equilateral triangle**, the **square**, and the **regular hexagon**.

- **Geometric Algebra**

In the second book of Euclid's *Elements*, there is a series of propositions that are essentially geometric expressions of algebraic formulas. A prime example is:

Proposition II.1:

"If there are two straight lines, and one of them is cut into any number of segments, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments."

This corresponds to the formula:

$$\alpha(\beta + \gamma + \dots) = \alpha\beta + \alpha\gamma + \dots$$

According to Eudemus (a Rhodian philosopher, 350–290 BC), a significant portion of Geometric Algebra is a Pythagorean discovery. The fact that Euclid had adopted various forms of systems of equations, along with their solutions, from cuneiform texts is a clear indication that the geometric algebra in his second book originates from Babylonian algebra. It is evident that the Pythagoreans formulated and geometrically proved the Babylonian rules used for solving these systems.

Astronomy – Laws of Numbers and the Universe – Cosmic Musical Harmony

In his work *On the Heavens* (*De Caelo*, II, 13), Aristotle writes that "the Pythagoreans placed fire at the center, and considered the Earth to be a star that moves in a circle around this central body." Later, in 270 BC, Aristarchus of Samos formulated the Copernican theory regarding the Earth and the Sun. Nicolaus Copernicus himself, who in the early 16th century AD proposed the theory that the Earth revolves around the Sun, considered his system —as did Galileo— to be essentially Pythagorean. It is evident that the study of numbers and proportions by Pythagoras and his school led to the concept of the cosmic, musical harmony of the universe.

Music

Pythagoras is credited with the theory of the functional significance of numbers within the world of music.

Potential Students of Pythagoras at the Samos School

47

γ) Πιθανοὶ μαθηταὶ τοῦ Πυθαγόρου ἐν Σάμῳ

1) Ἀβάρης ὁ Ὑπερβόρειος	29) Ἴππασος ὁ Ἀθηναῖος
2) Ἀνάχαρσις ὁ Σκύθης	30) Ἴππασος ὁ Λοκρὸς
3) Ἀριγνώτη ἡ Σαμία	31) Ἴππων ὁ Σάμιος
4) Ἀρίμνητος ὁ Σάμιος	32) Καλλίας
5) Ἀριστέας	33) Κλεόβουλος ἐκ Ῥόδου
6) Ἀριστοκράτης ὁ ἐκ Ῥηγίου	34) Κλεόμβροτος ἐκ Σπάρτης
7) Βροντίνος ὁ Κροτωνιάτης	35) Λάκων
8) Βροντίνος ὁ ἐκ Μεταποντίου	36) Λέων ὁ Φλειάσιος
9) Δαιμῶναξ ὁ Κυρηναῖος	37) Λεωνίδας ὁ Σπαρτιάτης
10) Δεινῶ σύζυγος Βροντίνου	38) Μέλισσα
11) Δημακίδης ὁ Ἰατρὸς	39) Μένων ὁ Κροτωνιάτης
12) Ἐκαταῖος ὁ Μιλήσιος	40) Μητροδωρος ὁ Κῶος
13) Ἐλοθαλῆς ὁ Κῶος	41) Μίλων ὁ Ὀλυμπιονίκης
14) Ἐλορίς ἡ Σαμία	42) Μιλτιάδης
15) Ἐπιμενίδης ὁ Κρής	43) Μυῖα
16) Εὐπαλίνιος	44) Ξενοφάνης
17) Εὐρυμένης Ὀλυμπιονίκης	45) Ὀκελλος ὁ ἐκ Λευκανίας
18) Ζάλευκος ὁ Λοκρὸς	46) Ὀλυμπιόδωρος
19) Ζάμολξις ὁ Θρᾷξ	47) Ὀλυμπιόδωρος
20) Ζώπυρος ὁ Κολοφώνιος	48) Παρμενίσκος
21) Ζώπυρος ὁ ἐκ Μεταποντίου	49) Περῶς ὁ ἐκ Κυρήνης
22) Θεανὼ θυγάτηρ Βροντίνου	50) Πυθαγόρας Διδάσκ. Ἀθλητῶν
23) Θεανὼ θυγάτηρ Πολυκράτους	51) Σπίνθαρος ὁ Κορίνθιος
24) Θεαίτητος ὁ ἐκ Ῥηγίου	52) Τιμάρατος ὁ Λοκρὸς
25) Θεόδωρος ὁ Σάμιος	53) Φερεκύδης ὁ ἐκ Δήλου
26) Θεοκλῆς ὁ ἐκ Ῥηγίου	54) Φύτιος ὁ ἐκ Ῥηγίου
27) Θέστωρ ὁ Ποσειδώνιος	55) Χαρώνδας ἐκ Κατάνης
28) Θύρσος ὁ Κῶος	56) Χιλωνίς ἐκ Σπάρτης

Georgios Sakellariou, Professor of Philosophy at the University of Athens, in his book *"PYTHAGORAS: The Teacher of the Centuries"* (Athens 1963, Ideotheatron Publications), provides a list of potential students of Pythagoras from his School in Samos.

Theano, Daughter of Brontinus

Theano was one of the foremost Pythagorean philosophers and is widely regarded as the most famous female astronomer and cosmologist of antiquity. She is credited with formulating the theory of the "**Golden Ratio**" (the Golden Mean) as well as the theory of the "**Harmony of the Spheres**."

Initially a student of Pythagoras, she later became his wife, despite their thirty-year age difference. She taught Astronomy and Mathematics at the Pythagorean Schools in Croton and, following her husband's death, continued her work in Samos.

The following works are attributed to Theano:

- *Pythagorean Aphorisms*
- *Advice to Women*
- *On Piety*
- *Philosophical Commentaries*
- *Letters*
- *The Life of Pythagoras*
- *Theory of Numbers*
- *The Structure of the Universe*

Arignote of Samos

The philosopher and mathematician Arignote was the daughter and student of Pythagoras. She authored the book "**On Numbers**" (**Περί Αριθμών**), though this work has not survived. Following the dissolution of the school in Croton, she returned to Samos.

Myia

Myia was the daughter of Pythagoras and Theano. Historical records mention her as an expert in **Geometry**. Scholars attribute to her the invention of the **third (or stationary) mean** (**εσθηκία μεσότης**). In this context, if we have the numbers α , β , γ , and the ratio $\alpha/\beta = \beta/\gamma$, holds true, then β represents the third proportion.

Deino

Deino was the wife of Brontinus. She was a student and the mother-in-law of Pythagoras, possessing deep knowledge of **Arithmosophy** (**αριθμοσοφία**). She studied **deficient numbers** (**ελλειπείς αριθμοί**). A number is deficient when the sum of its proper divisors (all divisors excluding the number itself) is less than the number itself. For instance, the number 8 is deficient because $1+2+4=7<8$.

Eloris of Samos

She also possessed knowledge of Geometry.

Melissa

A student of Pythagoras. She focused on the construction of regular polygons. Lobon of Argos (an ancient Greek historian and author of the 6th or 7th century BC) writes about an unknown work of hers: "The nature of the circle is the essence of all inscribed polygons" (**Ο ΚΥΚΛΟΣ ΦΥΣΙΝ ΤΩΝ ΕΓΓΡΑΦΟΜΕΝΩΝ ΠΟΛΥΓΩΝΩΝ ΑΠΑΝΤΩΝ ΕΣΤΙ**).

Hecataeus of Miletus

Son of Hegesander, born in Miletus (550-480 BC). Historiographer-logographer.

Eupalinos

Engineer and Architect from Megara. He constructed the famous Tunnel of Eupalinos (**Ευπαλίνειο ὄρυγμα**) in Pythagoreion, Samos, under a commission from Polycrates.

Theodorus of Samos

He lived in the 6th century BC. Sculptor and architect.

Hippon of Samos

Philosopher (6th/5th century BC).

Arimnestus of Samos

The son and student of Pythagoras. He became the teacher of Democritus of Abdera. Following his father's death, he departed from Croton and returned with his mother to Samos.

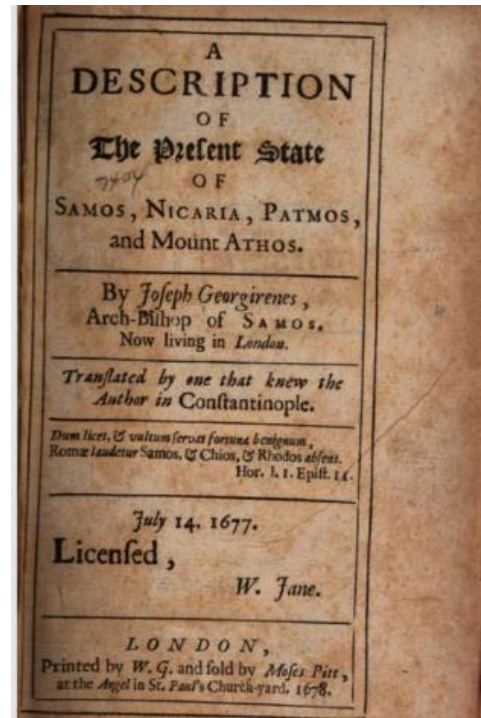
The Cave of Pythagoras in Samos

The Cave of Pythagoras, also known as the Cave of Panagia Sarantaskaliotissa, sits on the southeastern side of Mount Kerkis, 3 kilometers outside Marathokampos. It stands at an altitude of 350 meters above sea level and offers an impressive view that reaches the sea.

The nickname "Cave of Pythagoras" originates from the Samian philosopher and mathematician Pythagoras (580–495 BC). According to tradition, he either used the cave as a study or hid there while fleeing enemies provoked by the Milesian philosopher Anaximander (611–547 BC). Another version suggests he sought isolation there because he could not compromise with the regime of the tyrant Polycrates.

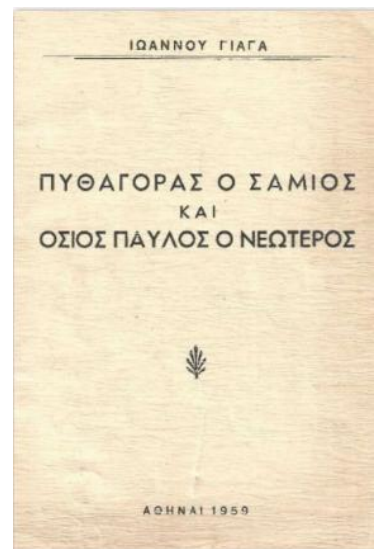
Regardless of the most likely version, the site exudes a mysterious aura and naturally carries various legends and traditions. Archbishop Joseph Georgirenes of Samos, in his work *"A Description of the Present State of Samos, Nicaria, Patmos, and Mount Athos,"* mentions that Pythagoras used the location as a place of study.

In his book *"Pythagoras,"* Edward Brin writes: "Immediately after his return, they say, he began teaching again. He spoke on a natural plateau, while he and some select students lived in caves." It appears they used the caves according to their needs, purpose, and the occasion. Therefore, Pythagoras most likely used the cave of Sarantaskaliotissa as a study, residence, and place of meditation rather than a hiding place. The fame this fact gained since ancient times reinforces this view. If he had used it only as a hideout for a short period, only a few trusted individuals would know about it. In contrast, its use as a study made it known to many more people, allowing it to survive as a tradition.



All the above make it clear that the tradition regarding Pythagoras's use of this cave comes from the depths of time. This tends to prove that we deal with a real historical event. Besides, tradition supports most of what we know about the life and work of this great Samian philosopher.

As bibliography and folk tradition suggest, Giannis Giagas concludes in his 1959 book, *"Pythagoras the Samian and Saint Paul the Younger,"* that Pythagoras used the Sarantaskaliotissa cave. He bases this conclusion on the fact that the cave provided water; a small basin inside collects water from stalactites before a dangerous, deep chasm begins. Furthermore, dense vegetation originally hid the cave, while its steep location made it inaccessible, secure, and spacious. Regardless of Pythagoras's specific reason for using the cave —whether for hiding, reflection, or as a study — Sarantaskaliotissa offers a comparative advantage because it served every possible need.



Perhaps, therefore, Pythagoras used the cave on Mount Kerkis as the "Semicircle" for his students for about 10 years before departing Samos for Southern Italy. We can say that Pythagoras used caves for his teaching, perhaps following the example of his teacher, Pherecydes, who also taught in a cave. If one observes the images below showing the current state of the "Cave of Pythagoras," the name "Semicircle" might have originated from the shape of the cave's mouth, which resembles a semicircle.



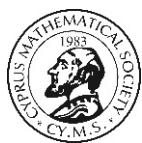
According to Iamblichus (XXVI), Pythagoras led his school for 39 years, spending 19 of those in

Samos and 20 in Croton. The cave offered easy access from the sea and sat at a great distance from the palace of Polycrates in present-day Pythagoreion.

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*Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026*

Pythagorean Thought and Artificial Intelligence: From Mathematical Harmony to Learning Algorithms

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ABSTRACT

The present article examines the profound philosophical and mathematical connection between the Pythagorean school of antiquity and modern Artificial Intelligence (AI). It investigates how the foundational principle of Pythagorean philosophy, that the essence of the world is based on numbers and mathematical ratios, serves as a precursor to computational thinking and learning algorithms. The article analyzes the historical evolution of this idea, from the Pythagoreans to the pioneers of computer science, and presents specific ways in which the Pythagorean concept of harmony and order is incorporated into the design and operation of modern AI systems. Finally, it highlights the ethical questions arising from both intellectual traditions, underscoring the timeless pursuit of knowledge, order, and responsibility.

Keywords: Pythagorean Philosophy, Artificial Intelligence, Mathematical Harmony, Algorithms, Pattern Recognition, Ethics.

Introduction: The Arithmetical Basis of Reality

In the 6th century BC, in Croton of Southern Italy, Pythagoras and his followers established a school of thought that would forever alter the course of Western civilization. The Pythagorean school was not merely a mathematical academy; it was a holistic philosophical, religious, and ethical community (Riedweg, 2005). The core of its teaching was a simple yet revolutionary idea: "All is number."

This dogmatic view held that the deepest essence of reality, the structure of the cosmos, and even ethics and aesthetics could be discerned and understood through mathematical relationships and proportions. The "Cosmic Harmony" or the "Harmony of the Spheres" was the manifestation of this mathematical order (Kahn, 2001). This perception was not just an abstract philosophical position. It was the cornerstone that paved the way for the development of geometry, number theory, music theory, and, ultimately, modern science (Izdebska, 2016).

In the modern era, we are experiencing another radical revolution: the rise of Artificial Intelligence (AI). AI, defined as the ability of a machine to replicate human cognitive functions such as learning and problem-solving, appears at first glance to be a product of the digital age (Mitchell, 2019). However, a careful examination reveals that its foundations are deeply rooted in the very same Pythagorean idea: that the world can be understood, modeled, and predicted through mathematical models and algorithms (Fasiku, 2024). The aim of this article is to systematically analyze this timeless bridge. The discussion that follows shows how Pythagorean thought was not merely a historical precursor but a theoretical background that continues to inform and inspire the development of Artificial Intelligence.

The Pythagorean Legacy: Foundations of Mathematical Thought

The Pythagorean school enforced a systematic study of mathematics, music, philosophy, and ethics (Riedweg, 2005). Their central finding; that mathematical laws govern nature, left an inexhaustible legacy. In geometry and numbers, the Pythagorean Theorem is the most famous example, but its influence was broader. The pursuit of symmetry, incommensurable numbers, and mathematical relations formed the basis for Euclid's "Elements" (Negrepontis, Farmaki, 2025). In music and harmony, the Pythagoreans discovered that musical consonances correspond to simple arithmetical ratios, proving that even aesthetic enjoyment had a mathematical basis, reinforcing the idea of Cosmic Harmony. Philosophically, Plato incorporated these ideas into his philosophy of the Forms, considering mathematical objects as the truest form of reality. This tradition passed through Archimedes, Diophantus, a forerunner of algebra, and reached Newton and Kepler, who sought the "laws of harmony" in planetary motion (Kirk, Raven, Schofield, 1999). This mathematical narrative of reality constituted the intellectual framework for the scientific revolution. Pythagoras, in this sense, can be characterized as a timeless "algorithmist of the universe" someone who believed the world operates based on predetermined, mathematical "algorithms."

The Rise of Artificial Intelligence: The Modern Pythagoreans

Artificial Intelligence, officially born at Dartmouth in 1956, is the heir to this tradition. While the Pythagoreans used logic and observation to reveal the mathematical laws of the universe, AI scientists

use algorithms and data to "learn" these laws from the environment. The core of every modern AI system is mathematical: Linear Algebra for representing and manipulating data; Calculus and Optimization for training models by minimizing a "cost function"; Probability Theory and Statistics for making inferences and handling uncertainty; and Discrete Mathematics and Logic for the structure of algorithms and knowledge representation (Holmes, 2024).

The analogy between the two paradigms is evident. The Pythagorean School relied on numbers and ratios to explain nature, while AI relies on algorithms and data to interpret the world. The goal of the former was the understanding of cosmic harmony, while the latter aims at pattern recognition and decision-making. In both, Philosophy and Mathematics are united, and both raise profound ethical questions about the application of knowledge (Govender, 2023).

Points of Convergence: Pythagorean Principles in Modern Algorithms

The Pythagorean legacy is not merely a historical curiosity. It manifests in many areas of modern AI. In image analysis and geometry, Computer Vision systems use algorithms that seek geometric patterns and symmetries to recognize objects (Floridi, 2014). This search for geometric logic is a direct descendant of Pythagorean geometry. In music composition and sound analysis, AI algorithms can compose music based on the very harmonic principles and mathematical ratios discovered by the Pythagoreans. In neural networks and pattern recognition, the learning process itself resembles a search for the most "harmonic" mathematical relationship that corresponds to the data, directly mirroring the Pythagorean quest for mathematical analogies in nature (Rowlett, 2024). Even clustering and classification algorithms, which aim to find structure and order in unstructured data, are a computational version of the Pythagorean belief that the universe is inherently orderly.

Ethics and Philosophy: A Timeless Dialogue

Both the Pythagorean school and Artificial Intelligence are concerned not only with technical knowledge but also with deeper philosophical and ethical questions. For the Pythagoreans, the study of mathematics and the pursuit of knowledge were inseparable from ethical life (Kahn, 2001). Understanding cosmic order led to inner harmony and a "pure" life. Knowledge was essential for ethical conduct (Burket, 1972). In the same spirit, the development of AI raises critical ethical questions (Mittelstadt, 2019). Who is responsible for the decisions of AI? How can we ensure that algorithmic systems are fair, unbiased, and transparent? How will AI affect human dignity and work? These discussions on 'AI Trustworthiness' are the modern incarnation of the Pythagorean notion that knowledge must lead to ethical application. AI, therefore, requires not only mathematical precision but also a philosophical framework that recalls

the need for balance, justice, and responsibility, concepts so central to the Pythagoreans (Russell, Norvig, 2021).

Historical Timeline: From Croton to ChatGPT

A chronological sequence reveals the unbroken connection. In 530 BC, the founding of the Pythagorean School laid the foundations of the arithmetical basis of reality (Huffman, 1993). During antiquity and the Middle Ages, Aristotelian logic and mathematics kept the idea alive (Horky, 2013). In the 17th century, Descartes dreamed of "universal mathematics" and Leibniz of a "Universal Characteristic" based on logic. In the 19th century, George Boole created the algebra of logic, providing mathematical substrate for digital computers (Grattan-Guinness, 2000). In the 1930s-1940s, Alan Turing theorized the concept of a "Thinking Machine." In 1956, the Dartmouth conference officially established the term "Artificial Intelligence." In 1997, Deep Blue defeated Kasparov, proving AI's superiority in pattern recognition within a rule-based domain. From 2010 onward, the explosion of Big Data and Deep Neural Networks enabled AI to recognize complex patterns in images and speech (O Neil, 2016). In the 2020s, the emergence of Generative AI, like ChatGPT, demonstrates that systems can generate coherent text and ideas, mimicking human cognitive processes, based solely on mathematical models and algorithms (Holmes, 2024). This is the culmination of the Pythagorean vision: a system that, through numbers, simulates knowledge and language.

Another dimension in which the Pythagorean legacy resonates within contemporary AI lies in the epistemological problem of abstraction. The Pythagoreans maintained that numbers do not merely *describe* the world but reveal its underlying structure; they serve as the bridge between the visible and the intelligible (Netz, 2004). In modern AI research, abstraction plays a similarly crucial role: neural networks form increasingly complex internal representations that are not directly observable but serve as the conceptual scaffolding through which the system interprets data (Goodfellow, Bengio, Courville, 2016). These hidden layers, often opaque even to their creators, operate as algorithmic analogues of the Pythagorean realm of mathematical forms, an intermediate domain where raw sensory inputs are transformed into structured meaning. These parallel raises important philosophical questions about the nature of representation: to what extent do AI abstractions mirror genuine features of the world, and to what extent are they merely computational artifacts (Bostom, 2014)? Just as ancient philosophers debated whether numerical forms possess independent reality, today's AI theorists confront the question of whether neural representations constitute true ontology or simply useful fiction (Chalmers, 2023). The dialogue between ancient metaphysics and modern machine learning thus converges on a shared inquiry: how do abstract structures mediate our access to truth?

Conclusion

The connection between Pythagorean philosophy and Artificial Intelligence is profound and substantive. Pythagoras and his followers proposed an ontological model: the world is mathematical (Zhud, 2012). Artificial Intelligence is the scientific and technological realization of this model. It is the tool with which we not only observe the mathematical harmony of the universe but also model it, predict it, and, to some extent, recreate it within our machines (Tegmark, 2017). Understanding this timeless connection is not merely an academic exercise. It provides us with a rich philosophical framework to face the challenges of the digital age. It reminds us that behind every algorithm, every neural network, and every AI decision lies the ancient quest for order, harmony, and meaning (Domingos, 2015). As we advance in developing ever more powerful technologies, the Pythagorean ideals of accuracy, balance, and ethical knowledge must remain our compass, ensuring that our artificial intelligence ultimately serves humanity and aligns with the harmony we have sought since antiquity (Rahwan, 2018).

The dialogue between Pythagorean philosophy and contemporary Artificial Intelligence reveals a surprisingly coherent intellectual continuum: both worldviews strive to uncover the hidden structures that govern reality. For the Pythagoreans, the essence of the universe was expressed through numerical relations and harmonic proportions (Dijksterhuis, 1987). For today's AI systems, the world is reinterpreted through data patterns, optimization rules, and computational representations. Although separated by more than two millennia, these approaches converge on a fundamental conviction: that order is discoverable, that complexity can be modeled, and that understanding arises from decoding the relationships underlying phenomena (Sutton, Barto, 2018).

In this light, the Pythagorean insight that *"all things are number"* resonates with the modern assumption that data and algorithms can approximate, simulate, and even predict human and natural behavior (Zhud, 2012). Machine learning models, whether neural networks or probabilistic systems, operate through abstract mathematical principles that parallel ancient notions of harmony and ratio (Pasquale, 2015). Just as Pythagoras used the lyre string and geometric forms to reveal proportional laws in nature, contemporary AI uses linear algebra, gradient descent, and large-scale computation to uncover patterns in language, sound, images, and social structures (Netz, 2004). The mathematical spirit is the same; only the technical instruments have changed.

At the same time, the juxtaposition between these two eras invites a deeper reflection on the philosophical implications of constructing an intelligible world from numbers. For the Pythagoreans, numerical harmony was not merely descriptive but prescriptive, it embodied a cosmic order grounded in metaphysics and ethics (Kahn, 2001). Their mathematics was inseparable from their moral worldview, linking intellectual clarity with spiritual cultivation (Zhud, 2012). In contrast, AI's mathematical vision is predominantly instrumental, optimized for efficiency, prediction, and performance (Govebder, 2023). This difference raises essential questions: Can a purely computational understanding of the world remain

ethically neutral? Should algorithms be guided by principles of harmony analogous to those envisioned by Pythagorean thought? And if so, what would a “harmonic” approach to AI development look like within modern societies?

The rapid integration of AI into daily life makes these questions increasingly urgent. As predictive models influence human decision-making, resource distribution, access to information, and even interpersonal relationships, the need for a broader philosophical perspective becomes clear (Druga, Williams, Breazeal, Resnick, 2019). Pythagorean philosophy reminds us that technical capability must be accompanied by moral reflection. Harmonic balance in ancient thought was not just a feature of numbers but a guiding principle for living well within a community (Burket, 1972). Translating this to the age of AI suggests that algorithmic systems should not simply maximize accuracy or efficiency but should also promote fairness, transparency, and human flourishing (Fasiku, 2024). The pursuit of mathematical harmony thus evolves into an ethical imperative for the design and governance of intelligent technologies.

Additionally, the Pythagorean legacy invites a reconsideration of human cognition in relation to artificial systems. Pythagoras believed that understanding numerical patterns allowed the human mind to align itself with the deeper logic of the cosmos (Izdebska, 2016). Today, as AI replicates and sometimes surpasses human abilities in pattern recognition, we confront new questions regarding the boundaries of human knowledge (Goodfellow, Bengio, Courville, 2016). Are algorithms merely tools for extending our intellectual reach, or do they represent a shift in the epistemological status of mathematics itself? When a neural network discovers structures in complex data, structures that even specialists cannot fully explain, does this reflect a new kind of mathematical intuition, or does it indicate a growing opacity in the systems we create?

These questions point to a profound philosophical tension. On one hand, AI embodies the triumph of mathematical methods, a realization of the Pythagorean dream that the universe can be understood through formal structures (Fasiku, 2024). On the other hand, the opacity and unpredictability of certain AI models challenge traditional notions of rational comprehension and scientific explanation (Russell, Norvig, 2021). The Pythagorean worldview sought clarity, order, and harmony; modern AI sometimes produces results that are accurate but obscure. The responsibility falls on contemporary thinkers, scientists, and policymakers to reconcile these tendencies, ensuring that the expanding power of algorithmic systems remains grounded in principles of intelligibility and accountability.

Ultimately, revisiting Pythagorean thought in the context of Artificial Intelligence serves as more than intellectual exercise. It illuminates a path toward a more reflective and humane technological future. The ancient pursuit of harmony can inform modern approaches to algorithm design, data ethics, and digital governance. The Pythagorean emphasis on balance, between the material and the spiritual, the measurable and the meaningful, offers a counterpoint to the reductive tendencies that sometimes

accompany technological progress (Negreponis, Farmaki, 2025). It suggests that AI should be developed not merely as a computational artifact but as part of a broader cultural and moral framework.

As we stand at the intersection of ancient philosophy and cutting-edge technology, the challenge is to craft systems that reflect the best of both worlds: the precision of modern mathematics and the wisdom of classical thought. The legacy of Pythagoras thus endures, not only in the equations that underpin algorithms but in the enduring human aspiration to align knowledge, ethics, and harmony. By integrating these perspectives, the future of Artificial Intelligence may become not just more powerful but more coherent, responsible, and attuned to the deeper patterns that shape our shared reality.

The historical connection linking Pythagorean philosophy to contemporary Artificial Intelligence unfolds across more than two and a half millennia and reveals a continuous intellectual trajectory grounded in the belief that mathematical structures underpin both nature and cognition (Fasiku, 2024).

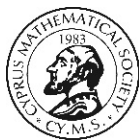
The origins of this trajectory can be traced to approximately 530 BC, when Pythagoras established his school in Croton and articulated the revolutionary claim that numerical relationships constitute the fundamental reality of the cosmos (Burkert, 1972). Through antiquity and the Middle Ages, this idea persisted in various philosophical and scientific traditions, particularly within Aristotelian logic, Neoplatonism, and medieval mathematical cosmology (Kirk, Raven, Schofield, 1999). During the seventeenth century, it resurfaced with renewed force: Descartes envisioned a universal mathematics capable of explaining every phenomenon through quantitative analysis, while Leibniz proposed a “Universal Characteristic,” a formal symbolic language intended to encode all human reasoning, foreshadowing the logical structures used in computation. The nineteenth century saw a decisive shift with George Boole’s algebra of logic, which transformed philosophical reasoning into mathematical operations and provided the conceptual foundation for modern digital computers (Rowlett, 2024). In the early twentieth century, Alan Turing extended this trajectory by formalizing the concept of computation through the Turing machine, establishing the theoretical possibility of mechanical reasoning. By 1956, the Dartmouth Conference introduced the term “Artificial Intelligence,” signaling the beginning of a scientific field dedicated to replicating aspects of human cognition through algorithms. Over the following decades, developments in symbolic AI, rule-based systems, and early neural networks paved the way for more sophisticated approaches. The late twentieth century culminated in 1997 with the computer Deep Blue defeating Garry Kasparov, a symbolic moment demonstrating that algorithmic systems could outperform humans in highly structured cognitive tasks. From 2010 onwards, the convergence of Big Data increased computational power, and the resurgence of deep neural networks enabled AI systems to identify complex patterns in images, language, and sensory data (O Neil, 2016). Most recently, the emergence of Generative AI in the 2020s, exemplified by models such as ChatGPT, has produced systems capable of generating coherent text, synthesizing information, and simulating aspects of reasoning through mathematical inference alone (Govender, 2023). This long intellectual timeline illustrates a continuous transformation of the Pythagorean claim “all is number” into the

contemporary paradigm in which intelligence itself is modelled, understood, and reproduced through numerical representation and algorithmic structure.

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*Mediterranean Journal for
Research in Mathematics Education
Vol. 21, 2026*

Pythagoras and the Transformation of Ancient Mathematical Thought

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ABSTRACT

This paper explores how Pythagoras of Samos transformed ancient mathematical knowledge into a rigorous and philosophical discipline. While earlier civilizations such as the Babylonians, Egyptians, Indians, and Chinese developed practical and often sophisticated mathematical systems, the use of mathematics remained largely empirical and utilitarian. Pythagoras introduced a revolutionary shift by viewing numbers not only as tools but also as metaphysical entities embodying the structure of reality. His work established mathematics as a fertile science, as evidenced by geometric proofs and his discoveries in number theory. More specifically, he unified music and mathematics through harmonic ratios, laying the foundations for mathematical aesthetics and cosmology. Despite challenges such as the discovery of irrational numbers, the Pythagorean tradition influenced Greek philosophy, especially through Plato, and shaped the trajectory of Western mathematical thought. This study examines the continuity and divergence between ancient traditions and Pythagorean innovations, highlighting Pythagoras's enduring role in the birth of abstract mathematical reasoning.

Key words: Pythagoras, Mathematics, Philosophy, Mathematical Reasoning, History of Mathematics

INTRODUCTION

Pythagoras of Samos (c. 570–495 BCE) is one of the most fascinating and elusive figures of antiquity. Although universally known for the mathematical theorem bearing his name, his life and teachings extended far beyond geometry - to music, cosmology, ethics, and metaphysics. His blend of science, mysticism, and philosophy left a lasting legacy, though often surrounded by myth and later embellishments.

Born on the island of Samos, Pythagoras was likely the son of Mnesarchus, possibly a merchant or craftsman. He received broad education in mathematics, philosophy, and astronomy, and later traveled to Egypt and perhaps Babylon, deepening his knowledge under the guidance of priests and sages. These journeys shaped his worldview and his belief in the divine nature of number and proportion.

Around 530 BCE, Pythagoras settled in Croton in southern Italy, where he founded a philosophical–religious community. This brotherhood functioned both as a school and a spiritual order, emphasizing ethical conduct, silence, vegetarianism, and the pursuit of purity. The group believed in metempsychosis -the transmigration of souls- and regarded numerical relationships as the key to understanding the universe.

Although Pythagoras is often credited with the geometric theorem that bears his name, the principle likely predates him. His greatest contribution was the philosophical elevation of mathematics. He famously linked music and mathematics through the study of vibrating strings, identifying simple ratios such as 2:1 (octave), 3:2 (fifth), and 4:3 (fourth) corresponding to harmonious sounds. These discoveries led him to believe that reality itself was structured by numerical relationships.

This belief culminated in the doctrine of the “Harmony of the Spheres.” According to this idea, planets emit musical tones as they move through space, creating a cosmic symphony inaudible to human ears but reflecting divine order and harmony. The concept influenced thinkers from Plato to Kepler and found a place in the history of cosmology and music theory.

Ancient accounts often depict Pythagoras as more than a philosopher; a divine figure or wonder-worker. Legends speak of his golden thigh, his ability to communicate with animals, and his healing powers. Though likely symbolic, these tales reveal the reverence his followers felt for him and highlight the mystical dimensions of his teachings.

His growing influence eventually provoked resistance. Anti-Pythagorean uprisings in Magna Graecia led to the destruction of their meeting places. Pythagoras is believed to have fled to Metapontum, where he died - though accounts vary as to whether his death was natural or violent.

Despite the collapse of his immediate circle, his teachings endured, profoundly influencing Plato and the Neoplatonists, and later reviving during the Renaissance through figures like Copernicus and Kepler. Kepler, in particular, drew directly on Pythagorean ideas in *Harmonices Mundi* (1619), seeking

geometric and harmonic order in planetary motion. In the modern era, Pythagorean thought continues to inspire theories in both music and physics

Pythagoras remains a symbolic figure of unity between reason and mysticism. Although much of his biography is shrouded in legend, his intellectual impact on mathematics, science, and spiritual thought is profound and enduring. He stands out as a philosopher who sought cosmic truth through numbers, harmony, and disciplined inquiry.

Pre-existing Knowledge

Pythagoras of Samos (c. 570–495 BCE) is considered a central figure in the history of mathematics not for inventing mathematical ideas from nothing, but for redefining and expanding them within a philosophical framework of lasting influence. Before Pythagoras, mathematical knowledge had developed primarily as a practical tool for commerce, agriculture, and astronomy.

Civilizations such as the Babylonians, Egyptians, Indians, and Chinese had developed rich traditions of arithmetic and geometry, including empirical methods for calculating areas, volumes, and ratios. However, these applications were largely empirical and lacked the abstract, deductive structure that characterizes mathematics as a scientific discipline today.

The Babylonians, for instance, had advanced numerical techniques, could solve quadratic equations, compute square roots, and use what we now call Pythagorean triples, such as the simple (3, 4, 5). Cuneiform tablets like Plimpton 322 (c. 1800 BCE) record sets of integers (a, b, c) satisfying $a^2 + b^2 = c^2$, indicating a deep understanding of right triangle relations, though given geometric interpretation only later. Their base-60 positional notation and practical algorithms for fractions and geometric measurement show sophistication but lacked the axiomatic rigor later introduced by the Greeks. In Egypt, mathematics served architecture, astronomy, and taxation. The Rhind Mathematical Papyrus (c. 1650 BCE) contains problems involving the areas of rectangles, triangles, and circles, and volumes of cylindrical granaries and pyramids. The 3:4:5 triangle, used to form right angles, was a tool of Egyptian surveyors measuring land after the Nile floods. Like the Babylonians, Egyptian mathematics remained focused on practical calculation, without generalization or proof-based reasoning.

Indian mathematics, preserved in texts such as the Sulba Sutras (c. 800–500 BCE), demonstrates advanced geometry in the context of Vedic ritual. These texts include verbal formulations of the Pythagorean Theorem and describe altar constructions requiring precise geometric design. The Baudhayana Sulba Sutra explicitly states that the diagonal of a rectangle produces an area equal to that of the two sides - a direct reference to the Pythagorean identity. Concepts of proportion and symmetry carried spiritual meaning, foreshadowing Pythagorean numerology.

China also had a rich mathematical tradition, most notably in *The Nine Chapters on the Mathematical Art* (Jiu Zhang Suan Shu), compiled over centuries and finalized during the Han dynasty (2nd century BCE). It included procedures for solving linear equations, calculating areas and volumes, and handling fractions. Early forms of Gaussian elimination appear in Chapter Eight, showing a matrix-like method for solving systems. The “Gougu Theorem,” equivalent to the Pythagorean Theorem, was demonstrated visually using

square diagrams. The Chinese approach, though algorithmic and empirical, emphasized practical clarity and precision.

These civilizations displayed remarkable mathematical ingenuity. However, Pythagoras and his school introduced a fundamentally different perspective: the abstract notion of mathematics as a universal principle of order, beauty, and truth. While others used numbers to describe and control the natural world, the Pythagoreans claimed that number itself was the essence of the world.

The Pythagorean Turn in Mathematics

The Greek innovation, particularly that of Pythagoras, lay in transforming procedural knowledge into theoretical systems grounded in analysis and logic. Pythagoras and his followers treated mathematics as both a philosophical and spiritual endeavor. Rather than a mere practical tool, number became the fundamental component of all reality.

This was a monumental shift -from empirical practice to metaphysical abstraction- summarized in the famous dictum: “All things are number.”

The enduring example of Pythagorean influence is, of course, the Pythagorean Theorem. Though known earlier, Pythagoras or his followers were the first to offer a general proof and integrate it into a deductive mathematical system, paving the way for Euclidean geometry.

The Pythagoreans also advanced number theory, categorizing numbers (odd and even, triangular, square, etc.) and investigating their properties systematically. They developed concepts like perfect and amicable numbers and established the early use of mathematical proof-central to Greek mathematics.

Even more revolutionary was the realization that numbers could model dynamic phenomena such as sound. Experiments with vibrating strings revealed that harmonious tones corresponded to small whole-number ratios. As Eli Maor notes, this discovery transformed mathematics from a discipline of quantity to one of quality, capable of describing aesthetic experiences such as musical harmony.

However, the discovery of irrational numbers-such as $\sqrt{2}$ -posed a crisis. The Pythagorean belief that “All is number” implied all quantities were expressible as ratios of integers. The emergence of incommensurable magnitudes undermined this worldview. According to Kitty Ferguson, such discoveries were suppressed within the school, illustrating the tension between idealism and empirical reality.

Nonetheless, this episode compelled later thinkers to confront deeper conceptual problems, advancing mathematical rigor. Euclid’s *Elements* (c. 300 BCE) drew heavily on Pythagorean methods, emphasizing axiomatic reasoning and logical deduction. Pythagorean numerical mysticism also influenced Plato’s *Theory of Forms* and the development of mathematical Platonism - the belief that mathematical objects exist in an abstract, nonmaterial realm accessible through reason.

Pythagoras's transformation, then, was not about inventing new techniques but about redefining mathematics as a means of understanding reality itself.

CONCLUSION

Pythagoras's legacy lies not in inventing mathematics but in transforming its purpose, scope, and philosophical meaning. Earlier civilizations achieved impressive empirical feats, but Pythagoras elevated mathematics from the practical to the abstract, from the utilitarian to the theoretical.

He redefined numbers as the fundamental principles of existence, establishing the foundation for deductive reasoning that would culminate in Euclidean geometry and shape Western thought through Plato and mathematical Platonism. His insights into musical harmony extended mathematics into aesthetics and cosmology, influencing fields from Kepler's celestial harmonies to modern wave theory and quantum mechanics.

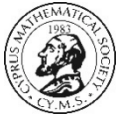
Though the school's discomfort with irrational numbers revealed its limitations, this crisis spurred progress toward greater rigor.

By reframing mathematics within a deeper philosophical and cosmic vision, Pythagoras introduced a new paradigm of abstraction and reasoning that continues to define scientific and philosophical inquiry. He transformed mathematics from a practical craft into a profound intellectual quest.

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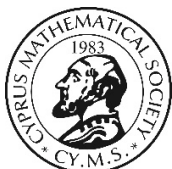
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